A CLASSIFICATION AND EXAMPLES
OF FOUR-DIMENSIONAL NONISOCLINIC
THREE-WEBS

Vladislav V. Goldberg

Abstract. A classification and examples of 4-dimensional nonisoclinic 3-webs of codimension 2 are given. The examples considered prove the existence for many classes of webs for which the general existence theorems are not proved yet.

0   Introduction

In the 1980s while studying rank problems for webs (see the papers [G 83, 92], the survey paper [AG 00] and the monograph [G 88], Ch. 8), the author has constructed three examples of exceptional 4-webs $W(4, 2, 2)$ of maximum 2-rank on a 4-dimensional manifold X^4 (see [G 85, 86, 87], [AG 00], and the books [G 88], [AS 92], and [AG 96]). They are exceptional since they are of maximum rank but not algebraizable. Hénaut [H 98] named these webs after the author and denoted them by $\mathcal{G}_1(4, 2, 2), \mathcal{G}_2(4, 2, 2)$, and $\mathcal{G}_3(4, 2, 2)$.

When the author was constructing these examples of 4-webs, he considered numerous examples of 3-webs $W(3, 2, 2)$ on X^4 and proved that 3 of them can be expanded to exceptional 4-webs $\mathcal{G}_1(4, 2, 2), \mathcal{G}_2(4, 2, 2)$, and $\mathcal{G}_3(4, 2, 2)$ (see [G 85, 86, 87, 88]).

However, it turns out that many examples of 3-webs $W(3, 2, 2)$ that the author has constructed in that research are useful when one studies different classes of multidimensional 3-webs (isoclinic, hexagonal, transversally geodesic, algebraizable, Bol's webs, etc.). Some of these examples of webs $W(3, 2, 2)$ were published in the author paper [G 92] and the book [G 88], and some were included in problem sections of the book [AS 92].

The author decided to present some of these examples both published and unpublished following some classification for them. It is worth to consider these examples in order to provide an up-to-date characterization of webs by indicating to which classes a web belongs.

For isoclinic webs this was done in [G 99]. In the current paper we present a classification and examples of nonisoclinic webs $W(3, 2, 2)$.

The examples mentioned above prove the existence of many classes of webs for which the general existence theorems are not proved yet.

1 The transversal distribution of a web $W(3, 2, 2)$

1. The leaves of the foliation λ_ξ, $\xi = 1, 2, 3$, of a web $W(3, 2, 2)$ are determined by the equations $\underset{\xi}{\omega^i} = 0$, $i = 1, 2$, where

$$\underset{1}{\omega^i} + \underset{2}{\omega^i} + \underset{3}{\omega^i} = 0 \tag{1}$$

(see, for example, [G 88], Section **8.1** or [AS 92], Section **1.3**).

The structure equations of such a web can be written in the form

$$\begin{cases} d\underset{1}{\omega^i} = \underset{1}{\omega^j} \wedge \omega_j^i + a_j \underset{1}{\omega^j} \wedge \underset{1}{\omega^i}, \\ d\underset{2}{\omega^i} = \underset{2}{\omega^j} \wedge \omega_j^i - a_j \underset{2}{\omega^j} \wedge \underset{2}{\omega^i}. \end{cases} \tag{2}$$

The differential prolongations of equations (2) are

$$d\omega_j^i - \omega_j^k \wedge \omega_k^i = b_{jkl}^i \underset{1}{\omega^k} \wedge \underset{2}{\omega^l}, \tag{3}$$

$$da_i - a_j \omega_i^j = p_{ij} \underset{1}{\omega^j} + q_{ij} \underset{2}{\omega^j}, \tag{4}$$

where

$$b_{[j|l|k]}^i = \delta_{[k}^i p_{j]l}, \quad b_{[jk]l}^i = \delta_{[k}^i q_{j]l} \tag{5}$$

(see [G 88], Sections **8.1** and **8.4** or [AS 92], Section **3.2**). The quantities

$$a_{jk}^i = a_{[j} \delta_{k]}^i \tag{6}$$

and b_{jkl}^i are the *torsion and curvature tensors* of a three-web $W(3, 2, 2)$. Note that for webs $W(3, 2, 2)$ the torsion tensor a_{jk}^i always has structure (6), where $a = \{a_1, a_2\}$ is a covector. If $a = 0$, then a web $W(3, 2, 2)$ is *isoclinicly geodesic*. In what follows in this paper, *we will assume that $a \neq 0$, i.e., a web $W(3, 2, 2)$ is nonisoclinicly geodesic.*

2. For a web $W(3, 2, 2)$, a transversally geodesic distribution is defined (cf. [AS 92], Section **3.1**) by the equations

$$\xi^2 \underset{1}{\omega^1} - \xi^1 \underset{1}{\omega^2} = 0, \quad \xi^2 \underset{2}{\omega^1} - \xi^1 \underset{2}{\omega^2} = 0.$$

If we take $\dfrac{\xi^1}{\xi^2} = -\dfrac{a_2}{a_1}$, we obtain an invariant transversal distribution Δ defined by the equations

$$a_1 \underset{1}{\omega^1} + a_2 \underset{1}{\omega^2} = 0, \quad a_1 \underset{2}{\omega^1} + a_2 \underset{2}{\omega^2} = 0. \tag{7}$$

We will call the distribution Δ defined by equations (7) the *transversal a-distribution* of a web $W(3, 2, 2)$ since it is defined by the covector a. Note that for isoclinicly geodesic webs $W(3, 2, 2)$, for which $a_1 = a_2 = 0$, the transversal distribution is not defined.

The following theorem gives the conditions of integrability of the distribution Δ (see the proofs of this and other results of this section in [AG 98]).

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Theorem 1 *The transversal a-distribution Δ defined by equations (7) is integrable if and only if the components a_1 and a_2 of the covector a and their Pfaffian derivatives p_{ij} and q_{ij} satisfy the conditions*

$$\begin{cases} a_2^2 p_{11} - 2a_1 a_2 p_{(12)} + a_1^2 p_{22} = 0, \\ a_2^2 q_{11} - 2a_1 a_2 q_{(12)} + a_1^2 q_{22} = 0. \end{cases} \tag{8}$$

3. For a web $W(3,2,2)$, it is always possible to take a specialized frame in which there is a relation between the components a_1 and a_2 of the covector a. For example, if the transversal distribution Δ coincides with the distribution $\underset{\alpha}{\omega}^1 = 0$, then we have $a_2 = 0$. In this case, the form ω_2^1 is expressed in terms of the basis forms $\underset{\alpha}{\omega}^i$, $\alpha = 1, 2$, i.e., in this case we have $\pi_2^1 = 0$, where $\pi_i^j = \omega_i^j \big|_{\underset{\alpha}{\omega}^i = 0}$.

In examples, that we are going to present in this paper, such situations will occur. So, we present here the conditions of integrability for the 4 cases that include the case indicated above.

Corollary 2 *If for a web $W(3,2,2)$ one of the following conditions*

$$a_2 = 0, \quad \pi_2^1 = 0, \tag{9}$$

$$a_1 = 0, \quad \pi_1^2 = 0, \tag{10}$$

$$a_1 = a_2, \quad \pi_1^1 + \pi_1^2 - \pi_2^1 - \pi_2^2 = 0, \tag{11}$$

$$a_1 = -a_2, \quad \pi_1^1 - \pi_1^2 + \pi_2^1 - \pi_2^2 = 0 \tag{12}$$

holds, then the a-distribution Δ coincides with the distribution $\underset{\alpha}{\omega}^1 = 0$, $\underset{\alpha}{\omega}^2 = 0$, $\underset{\alpha}{\omega}^1 + \underset{\alpha}{\omega}^2 = 0$, or $\underset{\alpha}{\omega}^1 - \underset{\alpha}{\omega}^2 = 0$, respectively. This a-distribution is integrable if and only if the quantities p_{ij} and q_{ij} satisfy respectively the following conditions:

$$p_{22} = q_{22} = 0, \tag{13}$$

$$p_{11} = q_{11} = 0, \tag{14}$$

$$p_{11} - 2p_{(12)} + p_{22} = 0, \quad q_{11} - 2q_{(12)} + q_{22} = 0, \tag{15}$$

$$p_{11} + 2p_{(12)} + p_{22} = 0, \quad q_{11} + 2q_{(12)} + q_{22} = 0. \tag{16}$$

Each of relations (8), (13), (14), (15), and (16) gives two conditions which Pfaffian derivatives p_{ij} and q_{ij} of the co-vector a must satisfy in order for the a-distribution Δ of a web $W(3,2,2)$ to be integrable.

4. Now we state the theorem giving the conditions for the integral surfaces of the a-distribution Δ to be geodesicly parallel in some affine connections.

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Theorem 3 *If on a web $W(3,2,2)$ the conditions*

$$\begin{cases} a_2p_{12} - a_1p_{22} = 0, & a_1p_{21} - a_2p_{11} = 0, \\ a_2q_{12} - a_1q_{22} = 0, & a_1q_{21} - a_2q_{11} = 0 \end{cases} \tag{17}$$

hold, then the integral surfaces V^2 of the a-distribution Δ are geodesicly parallel in any affine connection of the bundle of affine connections defined by the forms

$$\theta_v^u = \begin{pmatrix} \theta_j^i & 0 \\ 0 & \theta_j^i \end{pmatrix}, \quad i, j = 1, 2; \ u, v = 1, 2, 3, 4, \tag{18}$$

where

$$\theta_j^i = \omega_j^i + a_{jk}^i (p\underset{1}{\omega^k} + q\underset{2}{\omega^k}) \tag{19}$$

(see [AS 92], p. 35).

It follows from (4) that conditions (17) are equivalent to the following Pfaffian equation:

$$dt + t^2\omega_1^2 + t(\omega_1^1 - \omega_2^2) - \omega_2^1 = 0, \text{ where } t = \frac{a_2}{a_1}.$$

Note that it follows from Theorems 1 and 3 that if the surfaces V^2 of the a-distribution Δ are geodesicly parallel in any affine connection of the bundle (18)–(19), then the a-distribution Δ of a web $W(3,2,2)$ is integrable.

Corollary 4 *If the a-distribution Δ of a web $W(3,2,2)$ coincides with the distribution $\underset{\alpha}{\omega^1} = 0$, or $\underset{\alpha}{\omega^2} = 0$, or $\underset{\alpha}{\omega^1} + \underset{\alpha}{\omega^2} = 0$, or $\underset{\alpha}{\omega^1} - \underset{\alpha}{\omega^2} = 0$, then the integral surfaces V^2 of the a-distribution Δ are geodesicly parallel in any affine connection of the bundle (18)–(19) if and only if the quantities p_{ij} and q_{ij} satisfy respectively the following conditions:*

$$p_{2i} = q_{2i} = 0, \tag{20}$$

$$p_{1i} = q_{1i} = 0, \tag{21}$$

$$p_{1i} = p_{2i}, \ q_{1i} = q_{2i}, \tag{22}$$

$$p_{1i} = -p_{2i}, \ q_{1i} = -q_{2i}. \tag{23}$$

Note that conditions (20)–(23) are equivalent to the following relations between the forms ω_i^j (cf. the remark after Theorem 3): for these 4 possible specializations,

$$\begin{array}{ll} \omega_2^1 = 0, & \omega_1^2 = 0, \\ \omega_1^2 + \omega_1^1 - \omega_2^2 - \omega_2^1 = 0, & \omega_1^2 - \omega_1^1 + \omega_2^2 - \omega_2^1 = 0. \end{array}$$

5. Finally we state the following theorem.

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Theorem 5 *Let $W(3,2,2)$ be a web with nonvanishing covector $a \neq 0$ and with integrable transversal a-distribution Δ (conditions (8) hold). All 2-dimensional webs $W(3,2,1)$ cut by the foliations of a web $W(3,2,2)$ on the integral surfaces V^2 of Δ are hexagonal if and only if the equations*

$$-b^i_{111}a_2^3 + 3b^i_{(112)}a_2^2 a_1 - 3b^i_{(122)}a_2 a_1^2 + b^i_{222}a_1^3 = 0, \quad i = 1, 2, \qquad (24)$$

hold.

Corollary 6 *If the transversal a-distribution Δ of a web $W(3,2,2)$ is the distribution defined by the equations $\underset{\alpha}{\omega^1} = 0$, or $\underset{\alpha}{\omega^2} = 0$, or $\underset{\alpha}{\omega^1} + \underset{\alpha}{\omega^2} = 0$, or $\underset{\alpha}{\omega^1} - \underset{\alpha}{\omega^2} = 0$, then all 2-dimensional webs $W(3,2,1)$ cut by the foliations of a web $W(3,2,2)$ on the integral surfaces V^2 of Δ are hexagonal if and only if respectively the following equations*

$$b^1_{222} = 0, \quad b^2_{222} = 0, \qquad (25)$$

$$b^1_{111} = 0, \quad b^2_{111} = 0, \qquad (26)$$

$$\begin{cases} -b^1_{111} + 3(b^1_{(112)} - b^1_{(122)}) + b^1_{222} = 0, \\ -b^2_{111} + 3(b^2_{(112)} - b^2_{(122)}) + b^2_{222} = 0, \end{cases} \qquad (27)$$

$$\begin{cases} b^1_{111} + 3(b^1_{(112)} + b^1_{(122)}) + b^1_{222} = 0, \\ b^2_{111} + 3(b^2_{(112)} + b^2_{(122)}) + b^2_{222} = 0 \end{cases} \qquad (28)$$

hold.

2 Nonisoclinic webs $W(3,2,2)$

1. For such a web, we write equations (1)–(6) and their prolongations:

$$-\underset{3}{\omega^i} = \underset{1}{\omega^i} + \underset{2}{\omega^i}, \qquad (29)$$

$$\begin{cases} d\underset{1}{\omega^i} = \underset{1}{\omega^j} \wedge \omega^i_j + a_j \underset{1}{\omega^j} \wedge \underset{1}{\omega^i}, \\ d\underset{2}{\omega^i} = \underset{2}{\omega^j} \wedge \omega^i_j - a_j \underset{2}{\omega^j} \wedge \underset{2}{\omega^i}, \end{cases} \qquad (30)$$

$$d\omega^i_j - \omega^k_j \wedge \omega^i_k = b^i_{jkl} \underset{1}{\omega^k} \wedge \underset{2}{\omega^l}, \qquad (31)$$

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$$da_i - a_j\omega_i^j = \underset{1}{p_{ij}}\omega^j + \underset{2}{q_{ij}}\omega^j, \tag{32}$$

$$b^i_{[j|l|k]} = \delta^i_{[k}\underset{1}{p_{j]l}}, \quad b^i_{[jk]l} = \delta^i_{[k}\underset{2}{q_{j]l}}, \tag{33}$$

$$\nabla \underset{1}{p_{ij}} = \underset{1}{p_{ijk}}\underset{1}{\omega^k} + \underset{2}{p_{ijk}}\underset{2}{\omega^k}, \quad \nabla \underset{2}{q_{ij}} = \underset{1}{q_{ijk}}\underset{1}{\omega^k} + \underset{2}{q_{ijk}}\underset{2}{\omega^k}, \tag{34}$$

where

$$\nabla p_{ij} = dp_{ij} - p_{kj}\omega_i^k - p_{ik}\omega_j^k, \quad \nabla q_{ij} = dq_{ij} - q_{kj}\omega_i^k - q_{ik}\omega_j^k;$$

$$\begin{cases} \underset{1}{p_{i[jk]}} + p_{i[j}a_{k]} = 0, \quad \underset{2}{q_{i[jk]}} - q_{i[j}a_{k]} = 0, \\ \underset{2}{p_{ijk}} - \underset{2}{q_{ijk}} + a_m b^m_{ijk} = 0, \end{cases} \tag{35}$$

$$dp = p\omega_i^i + \underset{1}{p_i}\underset{1}{\omega^i} + \underset{2}{p_i}\underset{2}{\omega^i}, \quad dq = q\omega_i^i + \underset{1}{q_i}\underset{1}{\omega^i} + \underset{2}{q_i}\underset{2}{\omega^i}, \tag{36}$$

where

$$p_{[12]} = p \quad q_{[12]} = q, \quad \underset{k}{p_i} = \underset{k}{p_{[12]i}}, \quad \underset{k}{q_i} = \underset{k}{q_{[12]i}},$$

$$p^2 + q^2 > 0, \tag{37}$$

Recall that the condition (37) means that a web $W(3, 2, 2)$ is nonisoclinic.

2. The next theorem gives analytic characterizations for different types of nonisoclinic webs $W(3, 2, 2)$.

Theorem 7 *For nonisoclinic webs $W(3, 2, 2)$ of different types we have the following analytic characterizations:*

a) *A nonisoclinic web $W(3, 2, 2)$ is transversally geodesic if and only if*

$$b^i_{(jkl)} = \delta^i_{(j}b_{kl)}, \tag{38}$$

where b_{kl} is a $(0, 2)$-tensor.

b) *A nonisoclinic web $W(3, 2, 2)$ is hexagonal if and only if*

$$b^i_{(jkl)} = 0. \tag{39}$$

c) *A nonisoclinic web $W(3, 2, 2)$ is a Bol web B_m if and only if*

$$b^i_{j(kl)} = 0. \tag{40}$$

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d) *A nonisoclinic web $W(3,2,2)$ is a group web if and only if*

$$b^i_{jkl} = 0, \tag{41}$$

e) *A nonisoclinic three-web $W(3,2,2)$ can be uniquely extended to a nonisoclinic web $AGW(4,2,2)$ (see [G 88, 92]) if and only if the following conditions hold:*

$$p \neq 0, \; q \neq 0, \; p \neq q, \tag{42}$$

and

$$q(q\underset{1}{p}_i - p\underset{1}{q}_i) - p(q\underset{2}{p}_i - p\underset{2}{q}_i) = pq(p-q)a_i. \tag{43}$$

The 4th foliation of the web $AGW(4,2,2)$ is defined by the equations

$$p\underset{1}{\omega}^i + q\underset{2}{\omega}^i = 0. \tag{44}$$

Note that part a) of Theorem 7 implies the following two very practical tests for a web $W(3,2,2)$ to be nontransversally geodesic.

Corollary 8 a) *The nonvanishing of any of four components b^i_{jjj}, $i \neq j$, implies that a web $W(3,2,2)$ is not transversally geodesic.*

b) *If a web $W(3,2,2)$ is not a group 3-web and $b^1_{jkl} = 0$ or $b^2_{jkl} = 0$, then this web is not transversally geodesic.*

Proof. a) It follows from (38) that for a transversally geodesic $b^i_{jjj} = 0$, $i \neq j$.

b) Suppose that $b^2_{jkl} = 0$. Then it follows from (38) that $b_{jk} = 0$, and relation (38) implies that $b^1_{jkl} = 0$. As a result, the web $W(3,2,2)$ is a group 3-web. This contradicts to the conditions in b). ∎

We describe another practical test for a web $W(3,2,2)$ to be transversally geodesic or not. In general, to check whether a given web is transversally geodesic or nontransversally geodesic, one may assume that the web is transversally geodesic, calculate b_{ij} applying the formula $b_{ij} = \frac{3}{4}b^k_{(kij)}$ which is a consequence of (38), and substitute b_{ij} obtained into (38). If (38) will become the identity, the web in question is transversally geodesic, and if (38) fails, the web is nontransversally geodesic.

3. Suppose that 3 foliations λ_1, λ_2, and λ_3 of a web $W(3,2,2)$ are given as the level sets $u^i_\xi = \text{const.}$ ($\xi = 1,2,3$) of the following equations:

$$\lambda_1 : u^i_1 = x^i; \; \lambda_2 : u^i_2 = y^i; \; \lambda_3 : u^i_3 = f^i(x^j, y^k), \quad i,j,k = 1,2. \tag{45}$$

In order to characterize a web $W(3,2,2)$ given by (45), we must find the forms $\underset{\alpha}{\omega}^i$, $\alpha = 1,2$, ω^i_j, and the functions a^i_{jk}, b^i_{jkl}, a_i, $p_{ij}, q_{ij}, \underset{1}{p}_{ijk}, \underset{2}{p}_{ijk}, \underset{1}{q}_{ijk}$, $\underset{2}{q}_{ijk}, p, q, \underset{1}{p}_i, \underset{2}{p}_i, \underset{1}{q}_i$, and $\underset{2}{q}_i$.

The forms ω^i, ω^i_j and the functions a^i_{jk} and b^i_{jkl} can be found my means of the following formulas (see [AS 71], or [G 88], Section **8.1**, or [AS 92], Section **1.6**):

$$\omega^i_1 = \bar{f}^i_j dx^j, \quad \omega^i_2 = \tilde{f}^i_j dy^j, \quad \omega^i_3 = -du^i_3, \tag{46}$$

where

$$\bar{f}^i_j = \frac{\partial f^i}{\partial x^j}, \quad \tilde{f}^i_j = \frac{\partial f^i}{\partial y^j}, \quad \det(\bar{f}^i_j) \neq 0, \quad \det(\tilde{f}^i_j) \neq 0,$$

and

$$d\omega^i_1 = -d\omega^i_2 = \Gamma^i_{jk}\omega^j_1 \wedge \omega^k_2, \tag{47}$$

$$\Gamma^i_{jk} = -\frac{\partial^2 f^i}{\partial x^l \partial y^m}\bar{g}^l_j \tilde{g}^m_k, \tag{48}$$

$$\omega^i_j = \Gamma^i_{kj}\omega^k_1 + \Gamma^i_{jk}\omega^k_2, \tag{49}$$

$$a^i_{jk} = \Gamma^i_{[jk]}, \tag{50}$$

$$b^i_{jkl} = \frac{1}{2}\left(\frac{\partial \Gamma^i_{kl}}{\partial x^m}\bar{g}^m_j + \frac{\partial \Gamma^i_{jl}}{\partial x^m}\bar{g}^m_k - \frac{\partial \Gamma^i_{kj}}{\partial y^m}\tilde{g}^m_l - \frac{\partial \Gamma^i_{kl}}{\partial y^m}\tilde{g}^m_j\right.$$

$$\left. +\Gamma^m_{jl}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{ml} + 2\Gamma^m_{kl}a^i_{mj}\right). \tag{51}$$

As to the functions a_i, $p_{ij}, q_{ij}, \underset{1}{p}_{ijk}, \underset{2}{p}_{ijk}, \underset{1}{q}_{ijk}, \underset{2}{q}_{ijk}, p, q, \underset{1}{p}_i, \underset{2}{p}_i, \underset{1}{q}_i$, and $\underset{2}{q}_i$, they can be easily calculated from equations (30)–(36). Then we should check whether this web is nonisoclinic, that is, whether condition (37) holds. Following this, we can investigate to which of the classes indicated in Theorems 1, 3, 5, 7 the web in question belongs. In the case e) of Theorem 7, we can find equations (44) of the 4th foliation of the web $W(4,2,2)$, an extension of the 3-web in question. Integrating these equations, we will find closed-form equations of leaves of this 4th foliation.

In what follows, we will always assume that 3 foliations of a web $W(3,2,2)$ are given as follows:

$$\begin{cases} \lambda_1 : x^1 = \text{const.}, \quad x^2 = \text{const.}; \\ \lambda_2 : y^1 = \text{const.}, \quad y^2 = \text{const.}; \\ u^1_3 = f^1(x^j, y^k) = \text{const.}, \quad u^2_3 = f^2(x^j, y^k) = \text{const.} \end{cases} \tag{52}$$

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In examples of webs, that we are going to present in Section **3**, we will only specify the functions $f^1(x^j, y^k)$ and $f^2(x^j, y^k)$.

Note that it follows from (49) that all forms ω_j^i are expressed in terms of the forms $\underset{1}{\omega^i}$ and $\underset{2}{\omega^i}$ only. This means that the forms $\pi_j^i = \omega_j^i\big|_{\underset{\alpha}{\omega^i} = 0}$ vanish, $\pi_j^i = 0$. It follows that the 2nd conditions in equations (9)–(12) are always valid for webs defined by equations (52), and the meaning of equations (9)–(12) is that the transversal distribution Δ coincides with the distribution $\underset{1}{\omega^i} = 0$, $\underset{2}{\omega^i} = 0$, $\underset{1}{\omega^i} + \underset{2}{\omega^i} = 0$, $\underset{1}{\omega^i} - \underset{2}{\omega^i} = 0$, respectively.

4. We will present a classification of nonisoclinic webs $W(3, 2, 2)$ given by equations (52).

A. Webs with the integrable transversal distribution Δ.

    A$_1$. Webs with the integrable transversal distribution $a_1 \underset{1}{\omega^i} + a_2 \underset{2}{\omega^i} = 0$, $a_1, a_2 \neq 0$ ((8) holds).

        A$_{11}$. Webs for which the surfaces V^2 are geodesicly parallel ((17) holds).

        A$_{12}$. Webs foliated into 2-dimensional hexagonal webs $W(3, 2, 1)$ ((8) and (24) hold).

        A$_{13}$. Webs with the integrable transversal distribution $\underset{\alpha}{\omega^1} + \underset{\alpha}{\omega^2} = 0$ ((11) and (15) hold).

            A$_{131}$. Webs for which the surfaces V^2 are geodesicly parallel ((11) and (22) hold).

            A$_{132}$. Webs foliated into 2-dimensional hexagonal webs $W(3, 2, 1)$ ((11), (15) and (27) hold).

        A$_{14}$. Webs with the integrable transversal distribution $\underset{\alpha}{\omega^1} - \underset{\alpha}{\omega^2} = 0$ ((12) and (16) hold).

            A$_{141}$. Webs for which the surfaces V^2 are geodesicly parallel ((12) and (23) hold).

            A$_{142}$. Webs foliated into 2-dimensional hexagonal webs $W(3, 2, 1)$ ((12), (16), and (28) hold).

    A$_2$. Webs with the integrable transversal distribution $\underset{\alpha}{\omega^1} = 0$ ((9) and (13) hold).

        A$_{21}$. Webs for which the surfaces V^2 are geodesicly parallel ((9) and (20) hold).

        A$_{22}$. Webs foliated into 2-dimensional hexagonal webs $W(3, 2, 1)$ ((9), (13), and (25) hold).

    A$_3$. Webs with the integrable transversal distribution $\underset{\alpha}{\omega^2} = 0$ ((10) and (14) hold).

        A$_{31}$. Webs for which the surfaces V^2 are geodesicly parallel ((10) and (21) hold).

A₃₂. Webs foliated into 2-dimensional hexagonal webs $W(3, 2, 1)$ ((10), (14), and (26) hold).

B. Webs with nonintegrable transversal distribution Δ.

C. Nontransversally geodesic webs ((38) does not hold).

D. Transversally geodesic webs ((38) holds).

    D₁. Hexagonal webs ((39) holds).

        D₁₁. Bol webs ((40) holds).

        D₁₂. Group webs ((41) holds).

E. Webs with different relations among p_{ij} and q_{ij}.

For webs defined by equations (52), each component of the tensors p_{ij} and q_{ij} is an absolute invariant, and the vanishing any of these components distinguishes a class of 3-webs $W(3, 2, 2)$. Note that since we consider only nonisoclinic webs $W(3, 2, 2)$, the tensors p_{ij} and q_{ij} cannot be simultaneously symmetric. Note also that the conditions $a_2 = 0$, $\omega_2^1 = 0$ imply the conditions $p_{2i} = q_{2i} = 0$ (cf. (20)), and the conditions $a_1 = 0$, $\omega_1^2 = 0$ imply the conditions $p_{1i} = q_{1i} = 0$ (cf. (21)). We indicate some of classes of such webs.

    E₁. Webs with $a_1 \neq 0$, $a_2 \neq 0$, $a_1 \neq a_2$.

        E₁₁. Webs with $p_{i2} = q_{i1} = 0$.

            E₁₁₁. Webs with $p_{21} = -q_{12}$.

        E₁₂. Webs with $p_{12} = p_{21}$, $q_{11} = q_{22}$, $q_{12} = -q_{21}$.

        E₁₃. Webs with $p_{11} = p_{22} = 0$.

            E₁₃₁. Webs with $p_{12} = q_{12}$, $p_{21} = q_{21}$.

    E₂. Webs with $p_{2i} = q_{2i} = 0$.

        E₂₁. Webs with $q_{12} = 0$.

        E₂₂. Webs with $p_{1i} = 0$.

        E₂₃. Webs with $p_{12} = q_{12}$.

    E₃. Webs with $p_{1i} = q_{1i} = 0$.

        E₃₁. Webs with $q_{2j} = 0$.

        E₃₂. Webs with $p_{21} = q_{21}$.

            E₃₂₁. Webs with $p_{22} = 0$.

        E₃₃. Webs with $p_{21} = -q_{21}$.

F. Webs extendable to exceptional webs $W(4, 2, 2)$ of maximum 2-rank ((42) and (43) hold).

G. Webs nonextendable to exceptional webs $W(4, 2, 2)$ of maximum 2-rank ((42) or (43) does not hold).

G$_1$. Webs with $p = 0$, $q \neq 0$.

G$_2$. Webs with $q = 0$, $p \neq 0$.

G$_3$. Webs with $p = q \neq 0$.

G$_4$. Webs with $p \neq 0$, $q \neq 0$, $p \neq q$ and for which condition (43) does not hold.

Remark 9 The classification presented above is complete in the sense that any web $W(3, 2, 2)$ belongs to the class **A** or **B**, **C** or **D**, **F** or **G**, **E** or the class of webs with other or no connections among p_{ij} and q_{ij}.

Remark 10 It is easy to see a geometric meaning of the classes **G$_1$** $-$ **G$_3$** for which conditions (42) do not hold. By (44), the 4th foliation λ_4 defining a non-isoclinic web $W(4, 2, 2)$ coincides with the foliations λ_1, λ_2, and λ_3, respectively. As to the class **G$_4$**, the 4th foliation is well defined but not integrable.

Remark 11 It is easy to give characterizations of webs of the classes **E$_1$** $-$ **E$_3$**:

- Class **E$_{11}$**: for webs of this class, both components, a_1 and a_2, are covariantly constant on the 2-dimensional distribution defined by the equations $\underset{1}{\omega^1} = \underset{2}{\omega^2} = 0$.

  - Class **E$_{111}$**: for webs of this class, in addition to be covariantly constant on the 2-dimensional distribution defined by the equations $\underset{1}{\omega^1} = \underset{2}{\omega^2} = 0$, the components a_1 and a_2 satisfy the exterior quadratic equation $\nabla a_1 \wedge \underset{1}{\omega^1} = \nabla a_2 \wedge \underset{2}{\omega^2}$. On some instances (see, for example Example **11** below) this condition is necessary and sufficient for integrability of the distribution defined by the equation $a_1 \underset{1}{\omega^1} = a_2 \underset{2}{\omega^2}$.

- Class **E$_{12}$**: for webs of this class, the quantities a_1 and a_2 satisfy the following 3 exterior quartic equations:

$$
\begin{cases}
\nabla a_1 \wedge \underset{1}{\omega^1} \wedge \underset{2}{\omega^1} \wedge \underset{2}{\omega^2} = -\nabla a_2 \wedge \underset{1}{\omega^2} \wedge \underset{2}{\omega^1} \wedge \underset{2}{\omega^2}, \\
\nabla a_1 \wedge \underset{1}{\omega^1} \wedge \underset{1}{\omega^2} \wedge \underset{2}{\omega^2} = -\nabla a_2 \wedge \underset{1}{\omega^1} \wedge \underset{1}{\omega^2} \wedge \underset{2}{\omega^1}, \\
\nabla a_1 \wedge \underset{1}{\omega^1} \wedge \underset{1}{\omega^2} \wedge \underset{2}{\omega^1} = \nabla a_2 \wedge \underset{1}{\omega^1} \wedge \underset{1}{\omega^2} \wedge \underset{2}{\omega^2}.
\end{cases}
$$

- Class **E$_{13}$**: for webs of this class, the component a_1 is covariantly constant on the one-dimensional distribution defined by the equation $\underset{1}{\omega^2} = \underset{2}{\omega^i} = 0$, and the component a_2 is covariantly constant on the 2-dimensional distribution defined by the equations $\underset{2}{\omega^1} = \underset{2}{\omega^i} = 0$.

– Class $\mathbf{E_{131}}$: for webs of this class, both the components a_1 and a_2 are covariantly constant on the 2-dimensional distributions defined by the equations $\underset{2}{\omega}^1 = \underset{1}{\omega}^2 = \underset{2}{\omega}^2 = 0$ and $\underset{1}{\omega}^1 = \underset{1}{\omega}^1 = \underset{2}{\omega}^2 = 0$, respectively.

- Class $\mathbf{E_{21}}$: for webs of this class, the component a_1 is covariantly constant on the ome-dimensional distribution defined by the equations $\underset{1}{\omega}^i = \underset{2}{\omega}^1 = 0$.

- Class $\mathbf{E_{22}}$: for webs of this class, the component a_1 is covariantly constant on the 2-dimensional distribution defined by the equations $\underset{2}{\omega}^i = 0$.

- Class $\mathbf{E_{23}}$: for webs of this class, the component a_1 is covariantly constant on the one-dimensional distribution defined by the equations $\underset{1}{\omega}^1 = \underset{2}{\omega}^1 = \underset{1}{\omega}^2 + \underset{2}{\omega}^2 = 0$.

- Class $\mathbf{E_{31}}$: for webs of this class, the component a_1 is covariantly constant on the entire web, and the component a_2 is covariantly constant on the 2-dimensional distribution defined by the equations $\underset{1}{\omega}^i = 0$.

- Class $\mathbf{E_{32}}$: for webs of this class, the component a_1 is covariantly constant on the entire web, and the component a_2 is covariantly constant on the one-dimensional distribution defined by the equations $\underset{1}{\omega}^2 = \underset{2}{\omega}^2 = \underset{1}{\omega}^1 + \underset{2}{\omega}^1 = 0$.

  – Class $\mathbf{E_{321}}$: for webs of this class, the component a_2 is covariantly constant on the 2-dimensional distribution defined by the equations $\underset{2}{\omega}^2 = \underset{1}{\omega}^1 + \underset{2}{\omega}^1 = 0$.

Remark 12 If a web is given by equation (52), then the webs of the classes $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_{13}}, \mathbf{A_{14}}$ as well as webs of the classes $\mathbf{A_{11}}, \mathbf{A_{131}}, \mathbf{A_{141}}, \mathbf{A_{21}}, \mathbf{A_{31}}$ and the classes $\mathbf{A_{12}}, \mathbf{A_{132}}, \mathbf{A_{142}}, \mathbf{A_{22}}, \mathbf{A_{32}}$ could be equivalent one to another. They are different by a location of the integrable transversal distribution Δ. In fact, if there is no additional conditions on webs, then by transformations $x^i = \phi^i(x^j)$, $y^i = \psi^i(y^j)$, we can make the integrable transversal distributions Δ of any two of them to coincide. However, in our examples in Section **3**, we always have $\pi_i^j = 0$, and above mentioned specializations are impossible. In addition, in our examples, there will be additional conditions on webs, and in general, under the above mentioned transformation (if it would be possible), these additional conditions for the first transformed web do not coincide with the additional conditions for the 2nd web. These are the reasons that in our classification, we consider all above mentioned classes as different ones.

Remark 13 The classification presented above has some overlapping classes: for example, the classes $\mathbf{A_{131}}$ and $\mathbf{A_{141}}$ are subclasses of the class $\mathbf{A_{11}}$, the classes $\mathbf{A_{132}}$ and $\mathbf{A_{142}}$ are subclasses of the class $\mathbf{A_{12}}$, and the classes $\mathbf{A_{21}}$ and $\mathbf{A_{31}}$ are subclasses of the classes $\mathbf{E_2}$ and $\mathbf{E_3}$, respectively.

Remark 14 In general, we must prove the existence theorem for all the classes listed above. Such a theorem can be proved for a web of general kind of each class using the well-known Cartan's test or it can be proved by finding examples of webs of these classes. The examples of webs in Section **3** prove the existence of webs of most of the classes of our classification.

3 Examples of nonisoclinic webs $W(3, 2, 2)$

In subsections **1–5**, we will consider examples of nonisoclinic webs $W(3, 2, 2)$ of the classes **F** and $\mathbf{G_1} - \mathbf{G_4}$, and for each of them, we will indicate to which other classes it belongs.

1. An example of an extendable nonisoclinic web $W(3, 2, 2)$ (Class F). We start from an example of a nonisoclinic web $W(3, 2, 2)$ that can be extended to a nonisoclinic web $\mathcal{G}_3(4, 2, 2)$.

Example 1 When the author was constructing the exceptional web $\mathcal{G}_3(4, 2, 2)$ (see [G 87]; see also [G 88], Example **8.4.5**, p. 431; [AG 96], Example **5.5.3**, p. 201; [AG 00], Section **1.4**, p. 101; and [AS 92], Ch. 8, problem **6**, p. 307), he started from an example of a web $W(3, 2, 2)$ defined by the equations

$$u_3^1 = x^1 + y^1 + \frac{1}{2}(x^1)^2 y^2, \;\; u_3^2 = x^2 + y^2 - \frac{1}{2}x^1(y^2)^2 \tag{53}$$

in a domain $x^1 y^2 \neq \pm 1$, where $\Delta = 1 + x^1 y^2$.

Using (48)–(51) and (32)–(37), we find that

$$\Gamma_{12}^1 = -\frac{x^1}{\Delta(2 - \Delta)}, \;\; \Gamma_{12}^2 = \frac{y^2}{\Delta(2 - \Delta)}, \;\; \Gamma_{11}^1 = \Gamma_{2i}^1 = \Gamma_{22}^2 = \Gamma_{i1}^2 = 0, \tag{54}$$

$$\begin{cases} \omega_1^1 = -\dfrac{x^1}{\Delta(2 - \Delta)}\omega_2^2, & \omega_2^1 = -\dfrac{x^1}{\Delta(2 - \Delta)}\omega_1^1, \\[3mm] \omega_1^2 = \dfrac{y^2}{\Delta(2 - \Delta)}\omega_2^2, & \omega_2^2 = \dfrac{y^2}{\Delta(2 - \Delta)}\omega_1^1, \end{cases} \tag{55}$$

$$a_1 = \frac{y^2}{\Delta(2 - \Delta)}, \;\;\; a_2 = \frac{x^1}{\Delta(2 - \Delta)}, \tag{56}$$

$$\begin{cases} p_{11} = \dfrac{2(y^2)^2(\Delta - 1)}{\Delta^3(2 - \Delta)^2}, & p_{21} = \dfrac{\Delta^2 - 2\Delta + 2}{\Delta^3(2 - \Delta)^2}, \\[3mm] q_{22} = \dfrac{2(x^1)^2(\Delta - 1)}{\Delta^2(2 - \Delta)^3}, & q_{12} = \dfrac{\Delta^2 - 2\Delta + 2}{\Delta^2(2 - \Delta)^3}, \\[3mm] p_{i2} = q_{i1} = 0, \end{cases} \tag{57}$$

$$p = -\frac{\Delta^2 - 2\Delta + 2}{2\Delta^3(2 - \Delta)^2}, \;\;\; q = \frac{\Delta^2 - 2\Delta + 2}{2\Delta^2(2 - \Delta)^3}, \tag{58}$$

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$$
\begin{cases}
\underset{1}{p_1} = \dfrac{y^2(-\Delta^3 + 4\Delta^2 - 8\Delta + 6)}{\Delta^5(2-\Delta)^3}, & \underset{2}{p_2} = \dfrac{x^1(-\Delta^3 + 3\Delta^2 - 6\Delta + 4)}{\Delta^4(2-\Delta)^4}, \\[4mm]
\underset{1}{q_1} = \dfrac{y^2(\Delta^3 - 3\Delta^2 + 6\Delta - 4)}{\Delta^4(2-\Delta)^4}, & \underset{2}{q_2} = \dfrac{x^1(\Delta^3 - 2\Delta^2 + 4\Delta - 2)}{\Delta^3(2-\Delta)^5}, \\[4mm]
\underset{1}{p_2} = \underset{1}{q_2} = \underset{2}{p_1} = \underset{2}{q_1} = 0,
\end{cases}
\qquad (59)
$$

$$
\begin{cases}
b^1_{112} = -\dfrac{\Delta^2 - 2\Delta + 2}{\Delta^3(2-\Delta)^2}, & b^1_{111} = b^1_{12i} = 0, \\[4mm]
b^1_{212} = -\dfrac{2(x^1)^2(\Delta - 1)}{\Delta^2(2-\Delta)^3}, & b^1_{211} = b^1_{22i} = 0, \\[4mm]
b^2_{112} = -\dfrac{2(y^2)^2(\Delta - 1)}{\Delta^3(2-\Delta)^2}, & b^2_{111} = b^2_{121} = b^2_{122} = 0, \\[4mm]
b^2_{212} = -\dfrac{\Delta^2 - 2\Delta + 2}{\Delta^2(2-\Delta)^3}, & b^2_{211} = b^2_{22i} = 0.
\end{cases}
\qquad (60)
$$

It follows from (56) and (57) that the conditions (8) do not hold. Therefore, the a-distribution defined by equations (7) is not integrable, and the web (53) belongs to the class **B**.

It follows from (60) and (38) that the web (53) is not transversally geodesic (and consequently is neither hexagonal, nor Bol, nor group web). In fact, if (38) holds, then (60) gives $b^1_{111} = b_{11} = 0$ but it follows from (38) that $b_{11} = -\frac{1}{12}b^2_{112} \neq 0$. This contradiction proves that the web (53) belongs to the class **C**. By (57), the web (53) belongs to the class **E$_{11}$**.

It follows from (56), (58), and (59) that the conditions (42) and (43) are satisfied. Hence the equations (53) define a *nonisoclinic* 3-web $W(3,2,2)$ *that can be extended to a nonisoclinic* 4-web $\mathcal{G}_3(4,2,2)$. Thus the web (53) belongs to the class **F**.

As we proved in [AG 87] (see also [G 88], Section **8.4**), the leaves of the 4th foliation of the web $W(4,2,2)$ are defined as level sets of the following functions:

$$
u^1_4 = -x^1 + y^1 + \frac{1}{2}(x^1)^2 y^2, \quad u^2_4 = x^2 - y^2 - \frac{1}{2}x^1(y^2)^2.
\qquad (61)
$$

It is also shown in [AG 87] (see also [G 88], Section **8.4**) that the 4-web defined by (53) and (61) is of maximum 2-rank, and the only abelian 2-equation has the form

$$
2dx^1 \wedge dx^2 + 2dy^1 \wedge dy^2 - du^1_3 \wedge du^2_3 + du^1_4 \wedge du^2_4 = 0.
\qquad (62)
$$

Note that *the 4-web constructed in this example represents the first and only known example of a nonisoclinic web $W(4,2,2)$ of maximum 2-rank.*

2. Examples of nonextendable nonisoclinic webs $W(3,2,2)$ with $p = 0$, $q \neq 0$ (Class G$_1$).

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Example 2 Suppose that a web $W(3, 2, 2)$ is given by

$$u_3^1 = x^1 + y^1, \quad u_3^2 = x^1 y^1 + x^2 y^2 \tag{63}$$

in a domain $x^2 \neq 0$, $y^2 \neq 0$ (see [G 88], Example **8.1.28**, p. 392). In this case using (48)–(51) and (32)–(37), we find that

$$\Gamma_{11}^2 = \frac{y^1}{y^2} - 1, \;\; \Gamma_{21}^2 = \frac{1}{y^2}, \;\; \Gamma_{i2}^2 = \Gamma_{jk}^1 = 0, \tag{64}$$

$$\omega_1^2 = \left(\frac{y^1}{y^2} - 1 \right) (\underset{1}{\omega^1} + \underset{2}{\omega^1}) + \frac{1}{y^2} \underset{1}{\omega^2}, \;\; \omega_2^2 = \frac{1}{y^2} \underset{2}{\omega^1}, \;\; \omega_i^1 = 0, \tag{65}$$

$$a_1 = -\frac{1}{y^2}, \quad a_2 = 0, \tag{66}$$

$$p_{ij} = q_{2i} = 0, \;\; q_{11} = -\frac{x^1}{x^2 (y^2)^2}, \;\; q_{12} = -\frac{1}{x^2 (y^2)^2}, \tag{67}$$

$$p = 0, \quad q = -\frac{1}{2 x^2 (y^2)^2}, \tag{68}$$

$$\begin{cases} b_{jkl}^1 = 0, \;\; b_{112}^2 = \frac{y^1}{x^2 (y^2)^2}, \;\; b_{121}^2 = -\frac{x^1}{x^2 (y^2)^2}, \;\; b_{211}^2 = \frac{y^1 - x^1}{x^2 (y^2)^2}, \\ b_{111}^2 = -\frac{1}{y^2} - \frac{x^1 y^1}{x^2 (y^2)^2}, \;\; b_{122}^2 = b_{221}^2 = \frac{1}{x^2 (y^2)^2}, \;\; b_{2i2}^2 = 0. \end{cases} \tag{69}$$

Equations (67) and (68) show that the web (63) belongs to the classes $\mathbf{E_{22}}$ and $\mathbf{G_1}$. Since by (69), $b_{111}^2 \neq 0$, it follows from (38) that the web (63) is not transversally geodesic (and consequently is neither hexagonal, nor Bol, nor group web). Thus it belongs to the class $\mathbf{C}$. Equations (66), (67), (69), (9), (13), (20), and (25) show that this web belongs to the classes $\mathbf{A_{21}}$ and $\mathbf{A_{22}}$.

Example 3 Consider the web $W(3, 2, 2)$ given by

$$u_3^1 = x^1 y^1 - x^2 y^2, \quad u_3^2 = x^1 y^2 + x^2 y^1 \tag{70}$$

in a domain where y^1 and y^2 are not 0 simultaneously, and x^1 and x^2 are not 0 simultaneously (see [G 88], Example **8.1.26**, p. 391 or [G 92], p. 341). In this case using (48)–(51) and (32)–(37), we find that

$$\begin{cases} \Gamma_{11}^1 = \Gamma_{22}^1 = -\Gamma_{12}^2 = \Gamma_{21}^2 = -\dfrac{\alpha}{\Delta_1 \Delta_2}, \\ \Gamma_{12}^1 = -\Gamma_{21}^1 = \Gamma_{11}^2 = \Gamma_{22}^2 = \dfrac{\beta}{\Delta_1 \Delta_2}, \end{cases} \tag{71}$$

$$\begin{cases} \omega_1^1 = \dfrac{1}{\Delta_1\Delta_2}\big[-\alpha(\underset{1}{\omega^1} + \underset{2}{\omega^1}) + \beta(\underset{2}{\omega^2} - \underset{1}{\omega^2})\big], \\[2mm] \omega_2^1 = \dfrac{1}{\Delta_1\Delta_2}\big[\beta(\underset{1}{\omega^1} - \underset{2}{\omega^1}) - \alpha(\underset{1}{\omega^2} + \underset{2}{\omega^2})\big], \\[2mm] \omega_1^2 = \dfrac{1}{\Delta_1\Delta_2}\big[\beta(\underset{1}{\omega^1} + \underset{2}{\omega^1}) + \alpha(\underset{2}{\omega^2} - \underset{1}{\omega^2})\big], \\[2mm] \omega_2^2 = \dfrac{1}{\Delta_1\Delta_2}\big[\alpha(\underset{1}{\omega^1} - \underset{2}{\omega^1}) + \beta(\underset{1}{\omega^2} + \underset{2}{\omega^2})\big], \end{cases} \tag{72}$$

$$a_1 = \frac{2\alpha}{\Delta_1\Delta_2}, \quad a_2 = -\frac{2\beta}{\Delta_1\Delta_2}, \tag{73}$$

$$\begin{cases} p_{11} = \dfrac{4(y^2)^2}{\Delta_1\Delta_2^2}, \;\; p_{22} = \dfrac{4(x^1)^2}{\Delta_1\Delta_2^2}, \;\; q_{11} = q_{22} = \dfrac{4(y^2)^2}{\Delta_1^2\Delta_2}, \\[3mm] p_{12} = p_{21} = -\dfrac{4x^1x^2}{\Delta_1\Delta_2^2}, \qquad\qquad q_{12} = -q_{21} = -\dfrac{4y^1y^2}{\Delta_1^2\Delta_2}, \end{cases} \tag{74}$$

$$p = 0, \quad q = -\frac{4y^1y^2}{\Delta_1^2\Delta_2}, \tag{75}$$

$$\begin{cases} b_{111}^1 = \dfrac{2\beta u_3^2}{\Delta_1^2\Delta_2^2}, \;\; & b_{112}^1 = \dfrac{2x^1y^1}{\Delta_1\Delta_2^2}, \;\; & b_{121}^1 = -\dfrac{2y^1y^2}{\Delta_1^2\Delta_2}, \;\; & b_{211}^1 = 0, \\[3mm] b_{222}^1 = -\dfrac{2\beta u_3^1}{\Delta_1^2\Delta_2^2}, \;\; & b_{122}^1 = -\dfrac{1}{\Delta_1\Delta_2^2}, \;\; & b_{212}^1 = \dfrac{2(y^2)^2}{\Delta_1^2\Delta_2}, \;\; & b_{221}^1 = \dfrac{(x^1)^2 - (x^2)^2}{\Delta_1\Delta_2^2}, \\[3mm] b_{111}^2 = \dfrac{2\alpha u_3^2}{\Delta_1^2\Delta_2^2}, \;\; & b_{112}^2 = \dfrac{2(x^2)^2}{\Delta_1\Delta_2^2}, \;\; & b_{121}^2 = \dfrac{2x^2\gamma}{\Delta_1^2\Delta_2^2}, \;\; & b_{211}^2 = \dfrac{2x^1\rho}{\Delta_1^2\Delta_2^2}, \\[3mm] b_{222}^2 = -\dfrac{2\alpha u_3^1}{\Delta_1^2\Delta_2^2}, \;\; & b_{122}^2 = \dfrac{\alpha^2}{\Delta_1^2\Delta_2^2}, \;\; & b_{212}^2 = -\dfrac{2x^2\gamma}{\Delta_1^2\Delta_2^2}, \;\; & b_{221}^2 = \dfrac{2y^2\sigma}{\Delta_1^2\Delta_2^2}. \end{cases} \tag{76}$$

where

$$\begin{cases} \Delta_1 = (y^1)^2 + (y^2)^2, & \Delta_2 = (x^1)^2 + (x^2)^2, \\[1mm] \alpha = x^1y^1 + x^2y^2, & \beta = x^1y^2 - x^2y^1, \\[1mm] \gamma = x^2[(y^1)^2 - (y^2)^2] + 2x^1y^1y^2, & \rho = x^1[(y^1)^2 - (y^2)^2] + 2x^2y^1y^2, \\[1mm] \sigma = y^1[(x^1)^2 - (x^2)^2] + 2x^1x^2y^2. \end{cases}$$

It follows from (73) and (74) that

$$a_2^2 q_{11} - 2a_1 a_2 q_{(12)} + a_1^2 q_{22} = \frac{16\beta^2(y^2)^2}{\Delta_1^3\Delta_2^4} \neq 0,$$

i.e, the conditions (8) do not hold. Therefore, the a-distribution defined by equations (7) is not integrable, and the web (70) belongs to the class **B**.

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Since by (76), $b^1_{222} \neq 0$, the web (70) is not transversally geodesic. (and consequently is neither hexagonal, nor Bol, nor group web). So this web belongs to the class **C**. By (74) and (75), the web (70) belongs to the classes $\mathbf{E_{12}}$ and $\mathbf{G_1}$.

3. Example of a nonextendable nonisoclinic web $W(3, 2, 2)$ with $p \neq 0$, $q = 0$ (Class $\mathbf{G_2}$).

Example 4 A web $W(3, 2, 2)$ is given by the equations

$$u^1_3 = x^1 + y^1, \quad u^2_3 = (x^1)^2 y^2 + (x^2)^2 y^1 \tag{77}$$

in a domain of $\mathbf{R}^4$ where $x^1, x^2, y^1 \neq 0$.

In this case using (48)–(51) and (32)–(37), we find that

$$\Gamma^1_{jk} = \Gamma^2_{22} = 0, \ \ \Gamma^2_{11} = \frac{(y^2)^2}{(x^1)^2} + \frac{x^1 y^2}{x^2 y^1}, \ \ \Gamma^2_{12} = -\frac{1}{(x^1)^2}, \ \ \Gamma^2_{21} = -\frac{1}{2x^2 y^1}, \tag{78}$$

$$\begin{cases} \omega^1_i = 0, \ \ \omega^2_2 = -\dfrac{1}{(x^1)^2}\omega^1_1 - \dfrac{1}{2x^2 y^1}\omega^1_2, \\[3mm] \omega^2_1 = \Big(\dfrac{(x^2)^2}{(x^1)^2} + \dfrac{x^1 y^2}{x^2 y^1}\Big)(\omega^1_1 + \omega^1_2) - \dfrac{1}{2x^2 y^1}\omega^2_1 - \dfrac{1}{(x^1)^2}\omega^2_2, \end{cases} \tag{79}$$

$$a_1 = -\frac{1}{(x^1)^2} + \frac{1}{2x^2 y^1}, \quad a_2 = 0, \tag{80}$$

$$\begin{cases} p_{11} = \dfrac{2}{(x^1)^3} + \dfrac{x^1 y^2}{2(x^2)^3(y^1)^2}, \ \ p_{12} = -\dfrac{1}{4(x^2)^3(y^1)^2}, \\[3mm] q_{11} = -\dfrac{1}{2x^2(y^1)^2}, \ \ p_{2i} = q_{2i} = q_{12} = 0, \end{cases} \tag{81}$$

$$p = -\frac{1}{8(x^2)^3(y^1)^2}, \quad q = 0, \tag{82}$$

$$\begin{cases} b^1_{jkl} = b^2_{222} = b^2_{122} = b^2_{212} = 0, \ \ b^2_{221} = \dfrac{1}{2(x^2)^3(y^1)^2}, \\[3mm] b^2_{111} = \dfrac{2(y^2)^2(y^2 - x^1)}{(x^1)^4} + \dfrac{y^2(x^1 + y^2)}{x^1 x^2 y^1} + \dfrac{x^1 y^2(x^1 y^2 + (x^2)^2)}{(x^2)^3(y^1)^2}, \\[3mm] b^2_{112} = \dfrac{(2x^1 - 1)(4(x^2 y^1 - (x^1)^2))}{4(x^1)^4 x^2 y^1}, \\[3mm] b^2_{121} = -\dfrac{2(x^1)^2(x^1 y^2 - x^2 y^1) + (x^2)^2 y^1}{4(x^1)^2(x^2)^3(y^1)^2}. \end{cases} \tag{83}$$

It follows from (80)–(83) that equations (9), (13), (20), and (25) hold. Thus the web (77) belongs to the classes $\mathbf{A_{21}}$ and $\mathbf{A_{22}}$.

Since by (83), $b^1_{222} \neq 0$, the web (77) is not transversally geodesic (and consequently is neither hexagonal, nor Bol, nor group web). So this web belongs to the class **C**. By (81) and (82), the web (77) belongs to the classes $\mathbf{E_{21}}$ and $\mathbf{G_2}$.

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4. Examples of nonextendable nonisoclinic webs $W(3, 2, 2)$ with $p = q \neq 0$ (Class $\mathbf{G_3}$).

Example 5 A web $W(3, 2, 2)$ is given by the equations

$$u_3^1 = x^2 e^{x^1 y^1}, \quad u_3^2 = x^2 + y^2 \tag{84}$$

in a domain of $\mathbf{R}^4$ where $x^1, x^2, y^1 \neq 0$ (see [G 88], Example **8.1.27**, p. 391 or [G 92], p. 341 or [AS 92], Ch. 3, Problem **5**, p. 133)).

In this case using (48)–(51) and (32)–(37), we find that

$$\Gamma_{jk}^2 = \Gamma_{i2}^1 = 0, \;\; \Gamma_{11}^1 = -\frac{(1 + x^1 y^1)e^{-x^1 y^1}}{x^1 x^2 y^1}, \;\; \Gamma_{21}^1 = \frac{1}{x^1 x^2 y^1}, \tag{85}$$

$$\omega_1^1 = -\frac{(1 + x^1 y^1)e^{-x^1 y^1}}{x^1 x^2 y^1}\left(\underset{1}{\omega^1} + \underset{2}{\omega^1}\right) + \frac{1}{x^1 x^2 y^1}\underset{1}{\omega^2}, \;\; \omega_2^1 = \frac{1}{x^1 x^2 y^1}\underset{2}{\omega^1}, \tag{86}$$

$$a_1 = 0, \;\; a_2 = -\frac{1}{x^1 x^2 y^1}, \tag{87}$$

$$p_{1i} = q_{ij} = 0, \;\; p_{22} = -\frac{1 + x^1 y^1}{(x^1 x^2 y^1)^2}, \;\; p_{21} = q_{21} = -\frac{e^{-x^1 y^1}}{(x^1 x^2 y^1)^2}, \tag{88}$$

$$p = q = \frac{e^{-x^1 y^1}}{2(x^1 x^2 y^1)^2}, \tag{89}$$

$$\begin{cases} b_{jkl}^2 = b_{jjj}^1 = b_{112}^1 = b_{12i}^1 = 0, \\ b_{211}^1 = -\dfrac{e^{-x^1 y^1}}{(x^1 x^2 y^1)^2}, \;\; 2b_{212}^1 = b_{221}^1 = \dfrac{1 - x^1 y^1}{(x^1 x^2 y^1)^2}. \end{cases} \tag{90}$$

It follows from (87)–(90) that equations (10), (14), (21), and (26) hold. Thus the web (84) belongs to the classes $\mathbf{A_{31}}$ and $\mathbf{A_{32}}$.

Suppose that the web (84) is transversally geodesic. Then (38) implies that $b_{kl} = \frac{3}{4}b_{(ikl)}^i$. In particular, by (90), we must have $b_{12} = \frac{1}{4}b_{211}^1$. Substituting this value of b_{12} into (38), we find that $b_{(112)}^1 = \frac{1}{6}b_{211}^1$. But it follows from (90) that $b_{(112)}^1 = b_{211}^1$. Since $b_{211}^1 \neq 0$ (see (90)), the web (84) is not transversally geodesic (and consequently is neither hexagonal, nor Bol, nor group web). So this web belongs to the class $\mathbf{C}$. By (88) and (89), the web (84) belongs to the classes $\mathbf{E_{31}}$ and $\mathbf{G_3}$.

Example 6 Suppose that a web $W(3, 2, 2)$ is given by the equations

$$u_3^1 = x^1 + y^1, \quad u_3^2 = -x^1 y^1 + x^2 y^2 \tag{91}$$

in a domain $x^2 \neq 0, y^2 \neq 0$ (see [G 88], Example 8.1.28, p. 392, [AS 92], Ch. **3**, Problem **4**, p. 133, [G 92], p. 341). In this case using (48)–(51) and (32)–(37), we find that

$$\Gamma_{jk}^1 = 0, \ \ \Gamma_{11}^2 = 1 - \frac{x^1 y^1}{x^2 y^2}, \ \ \Gamma_{12}^2 = -\frac{y^1}{x^2 y^2}, \ \ \Gamma_{21}^2 = -\frac{x^1}{x^2 y^2}, \ \ \Gamma_{22}^2 = -\frac{1}{x^2 y^2}, \tag{92}$$

$$\begin{cases} \omega_1^2 = \left(1 - \dfrac{x^1 y^1}{x^2 y^2}\right)\left(\omega_1^1 + \omega_2^1\right) - \dfrac{x^1}{x^2 y^2}\omega_1^2 - \dfrac{y^1}{x^2 y^2}\omega_2^2, \\[2mm] \omega_2^2 = -\dfrac{1}{x^2 y^2}\left(\omega_1^2 + \omega_2^2\right) - \dfrac{y^1}{x^2 y^2}\omega_1^1 - \dfrac{x^1}{x^2 y^2}\omega_2^1, \ \ \omega_i^1 = 0, \end{cases} \tag{93}$$

$$a_1 = \frac{x^1 - y^1}{x^2 y^2}, \quad a_2 = 0, \tag{94}$$

$$\begin{cases} p_{2i} = q_{2i} = 0, & p_{12} = q_{12} = \dfrac{y^1 - x^1}{(x^2 y^2)^2}, \\[2mm] p_{11} = \dfrac{x^2 y^1 - x^1 y^2 + (y^1)^2}{(x^2 y^2)^2}, & q_{11} = \dfrac{-x^2 y^1 + x^1 y^2 - (x^1)^2}{(x^2 y^2)^2}, \end{cases} \tag{95}$$

$$p = q = \frac{y^1 - x^1}{2(x^2 y^2)^2} \tag{96}$$

$$\begin{cases} b_{jkl}^1 = b_{222}^2 = b_{21i}^2 = 0, \\[2mm] b_{111}^2 = \dfrac{(x^1 - y^1)(x^2 y^2 - x^1 y^1)}{(x^2 y^2)^2}, \ \ b_{112}^2 = \dfrac{(y^1)^2 - x^1 y^1 + x^2 y^2}{(x^2 y^2)^2}, \\[2mm] b_{121}^2 = \dfrac{-(x^1)^2 + x^1 y^1 - x^2 y^2}{(x^2 y^2)^2}, \ \ b_{122}^2 = 2b_{221}^2 = \dfrac{y^1 - x^1}{(x^2 y^2)^2}. \end{cases} \tag{97}$$

It follows from (94)–(97) that equations (9), (13), (20), and (25) hold. Thus the web (91) belongs to the classes $\mathbf{A_{21}}$ and $\mathbf{A_{22}}$.

Since by (97), $b_{111}^2 \neq 0$, the web (91) is not transversally geodesic. So this web belongs to the class $\mathbf{C}$. By (95) and (96), the web (91) belongs to the classes $\mathbf{E_{23}}$ and $\mathbf{G_3}$.

Example 7 Suppose that a web $W(3, 2, 2)$ is given by the equations

$$u_3^1 = x^1 y^1 + x^2 (y^2)^2, \quad u_3^2 = x^2 + y^2 \tag{98}$$

in a domain $x^1 \neq 0$, $y^1 \neq 0$. In this case using (48)–(51) and (32)–(37), we find that

$$\begin{cases} \Gamma^2_{jk} = 0, \ \ \Gamma^1_{11} = -\dfrac{1}{x^1 y^1}, \ \ \Gamma^1_{12} = \dfrac{2x^2 y^2}{x^1 y^1}, \\[2mm] \Gamma^1_{21} = \dfrac{(y^2)^2}{x^1 y^1}, \ \ \Gamma^1_{22} = -\dfrac{2y^2(x^2(y^2)^2 + x^1 y^1)}{x^1 y^1}, \end{cases} \tag{99}$$

$$\begin{cases} \omega^1_1 = -\dfrac{1}{x^1 y^1}\Big(\underset{1}{\omega^1} + \underset{2}{\omega^1}\Big) + \dfrac{(y^2)^2}{x^1 y^1}\underset{1}{\omega^2} + \dfrac{2x^2 y^2}{x^1 y^1}\underset{2}{\omega^2}, \ \ \omega^2_i = 0, \\[2mm] \omega^1_2 = -\dfrac{2x^2 y^2}{x^1 y^1}\Big(\underset{1}{\omega^1} + \underset{2}{\omega^2}\Big) + \dfrac{2}{x^1 y^1}\underset{1}{\omega^1} + \dfrac{(y^2)^2}{x^1 y^1}\underset{2}{\omega^1}, \end{cases} \tag{100}$$

$$a_1 = 0, \ \ a_2 = \dfrac{y^2(-2x^2 + y^2)}{x^1 y^1}, \tag{101}$$

$$\begin{cases} p_{1i} = q_{1i} = 0, & p_{22} = \dfrac{(y^2)^2[(y^2)^2 - 2x^1 y^2 - 2x^1 y^1]}{(x^1 y^1)^2}, \\[2mm] p_{21} = q_{21} = \dfrac{(2x^2 - y^2)y^2}{(x^1 y^1)^2}, & q_{22} = \dfrac{2x^2(y^2)^2(y^2 - 2x^2)}{(x^1 y^1)^2}, \end{cases} \tag{102}$$

$$p = q = -\dfrac{1}{p_{21}}, \tag{103}$$

$$\begin{cases} b^2_{jkl} = b^1_{1kl} = b^1_{211} = 0, & b^1_{222} = \dfrac{2u^1_3[(y^2)^2(2x^2 - y^2) + x^1 y^1]}{(x^1 y^1)^2} \\[2mm] b^1_{212} = \dfrac{2x^2(y^2)^2(y^2 - 2x^2)}{(x^1 y^1)^2}, & b^1_{221} = \dfrac{(y^2)^3(y^2 - 2x^2) - 2x^1 y^1 y^2}{(x^1 y^1)^2}. \end{cases} \tag{104}$$

It follows from (101)–(104) that equations (10), (14), (21), and (26) hold. Thus the web (98) belongs to the classes $\mathbf{A_{31}}$ and $\mathbf{A_{32}}$.

The web (98) is not transversally geodesic since $b^1_{222} \neq 0$ (see (104)). Thus the web (98) belongs to the class $\mathbf{C}$. By (102) and (103), the web (98) belongs to the classes $\mathbf{E_{32}}$ and $\mathbf{G_3}$.

Example 8 *Polynomial* 3-*webs.* We will call a web $W(3, 2, 2)$ *polynomial* if it is defined by the equations

$$u^i_3 = x^i + y^i + c^i_{jk} x^j y^k, \tag{105}$$

where $c^i_{jk} = $ const. and

$$\begin{aligned} \Delta_1 &= (1 + c^1_{1i} y^i)(1 + c^2_{2j} y^j) - c^2_{1i} c^1_{2j} y^i y^j \neq 0, \\ \Delta_2 &= (1 + c^1_{i1} x^i)(1 + c^2_{j2} x^j) - c^2_{i1} c^1_{j2} x^i x^j \neq 0 \end{aligned}$$

(see [G 88], Example **8.4.4**, p. 430).

For a polynomial web, using (48), (50) and (6), we obtain

$$
\begin{cases}
a_1 = \dfrac{1}{\Delta_1 \Delta_2}\Big[c_{21}^2 - c_{12}^2 + (c_{11}^2 c_{22}^2 - c_{12}^2 c_{21}^2)(y^1 - x^1) \\
\qquad\quad + (c_{12}^1 c_{11}^2 - c_{11}^1 c_{12}^2)x^1 + (c_{22}^1 c_{11}^2 - c_{21}^1 c_{12}^2)x^2 \\
\qquad\quad + (c_{11}^1 c_{21}^2 - c_{21}^1 c_{11}^2)y^1 + (c_{12}^1 c_{21}^2 - c_{22}^1 c_{11}^2)y^2 \Big], \\[2mm]
a_2 = \dfrac{1}{\Delta_1 \Delta_2}\Big[c_{12}^1 - c_{21}^1 + (c_{11}^1 c_{22}^1 - c_{12}^1 c_{21}^1)(y^2 - x^2) \\
\qquad\quad + (c_{22}^1 c_{11}^2 - c_{21}^1 c_{12}^2)x^1 + (c_{22}^1 c_{21}^2 - c_{21}^1 c_{22}^2)x^2 \\
\qquad\quad + (c_{12}^1 c_{21}^2 - c_{22}^1 c_{11}^2)y^1 + (c_{12}^1 c_{22}^2 - c_{22}^1 c_{12}^2)y^2 \Big].
\end{cases}
\tag{106}
$$

Differentiating equations (106) and applying (4), we can calculate the values of p_{ij} and q_{ij}. Their expressions obtained by a lengthy calculations are rather long. However, it appeared that

$$
p_{12} - p_{21} = q_{12} - q_{21} = \frac{1}{\Delta_1^2 \Delta_2^2}\Big[(c_{11}^1 + c_{21}^2)(c_{12}^1 - c_{21}^1) - (c_{22}^2 + c_{12}^1)(c_{21}^2 - c_{12}^2) \Big],
$$

i.e., we have

$$
p = q = \frac{1}{2\Delta_1^2 \Delta_2^2}\Big[(c_{11}^1 + c_{21}^2)(c_{12}^1 - c_{21}^1) - (c_{22}^2 + c_{12}^1)(c_{21}^2 - c_{12}^2) \Big].
\tag{107}
$$

Equation (107) shows that the same condition

$$
(c_{22}^2 + c_{12}^1)(c_{12}^2 - c_{21}^2) + (c_{11}^1 + c_{21}^2)(c_{12}^1 - c_{21}^1) \neq 0
\tag{108}
$$

is sufficient for both, $p \neq 0$ and $q \neq 0$. Therefore, a polynomial web (105) satisfying condition (108) is nonisoclinic.

As a result, the polynomial 3-webs (105) cannot be extended to a nonisoclinic $W(4,2,2)$, and hence it belongs to the class $\mathbf{G_3}$.

Note that condition (108) will not be satisfied for example if one of the following conditions holds:

- $c_{12}^1 = c_{21}^1, \;\; c_{12}^2 = c_{21}^2$.

- $c_{22}^2 = -c_{12}^1, \;\; c_{21}^2 = -c_{11}^1$.

- $c_{22}^2 = -c_{21}^1, \;\; c_{12}^2 = -c_{11}^1$.

- $c_{22}^2 = -c_{12}^1 = -c_{21}^2$.

- $c_{11}^1 = -c_{12}^2 = -c_{21}^2$.

In these cases further investigation is necessary to determine whether a web in question is isoclinic or nonisoclinic, and if it is nonisoclinic, to which of the classes $\mathbf{G_1} - \mathbf{G_4}$ it belongs.

Example 9 Take a particular case of the web (105) by taking $c^1_{jk} = 0$:

$$u^1_3 = x^1 + y^1, \quad u^2_3 = x^2 + y^2 + c^2_{jk}x^j y^k, \quad c^2_{jk} = \text{const.} \qquad (109)$$

where

$$\Delta_1 = 1 + c^2_{21}y^1 + c^2_{22}y^2 \neq 0, \quad \Delta_2 = 2 + c^2_{12}x^1 + c^2_{22}x^2 \neq 0.$$

In this case using (48)–(51) and (32)–(37), we find that

$$\begin{cases} \Gamma^1_{jk} = 0, \ \Gamma^2_{22} = \dfrac{c^2_{22}}{\Delta_1\Delta_2}, \ \Gamma^2_{12} = \dfrac{\alpha y^1 - c^2_{12}}{\Delta_1\Delta_2}, \ \Gamma^2_{21} = \dfrac{\alpha x^1 - c^2_{21}}{\Delta_1\Delta_2}, \\[2mm] \Gamma^2_{11} = -\dfrac{\Delta_1(\alpha x^2 + c^2_{11}) + (c^2_{11}y^1 + c^2_{12}y^2)(\alpha x^1 - c^2_{21})}{\Delta_1\Delta_2}, \end{cases} \qquad (110)$$

$$a_1 = -\frac{\beta}{\Delta_1\Delta_2}, \quad a_2 = 0, \qquad (111)$$

$$\begin{cases} p_{2i} = q_{2i} = 0, & p_{11} = \dfrac{-\alpha\Delta^2_1\Delta_2 + \beta(c^2_{12} - \alpha y^1)}{\Delta^3_1\Delta^2_2}, \\[2mm] p_{12} = q_{12} = \dfrac{c^2_{22}\beta}{\Delta^2_1\Delta^2_2}, & q_{11} = \dfrac{\alpha\Delta_1\Delta^2_2 + \beta(c^2_{21} - \alpha x^1)}{\Delta^2_1\Delta^3_2}, \end{cases} \qquad (112)$$

$$p = q = \frac{c^2_{22}\beta}{2\Delta^2_1\Delta^2_2}, \qquad (113)$$

where

$$\alpha = c^2_{11}c^2_{22} - c^2_{12}c^2_{21}, \quad \beta = \alpha(x^1 - y^1) + c^2_{12} - c^2_{21}.$$

In this case the condition (108) becomes

$$c^2_{22} \neq 0.$$

Thus for this web $p \neq 0$ and $q \neq 0$ if $c^2_{22} \neq 0$. Conditions (112) and (113) prove that the web (109) belongs to the classes $\mathbf{E_{23}}$ and $\mathbf{G_3}$. Note that the condition $c^2_{22} = 0$ is necessary and sufficient for the web (109) to be isoclinic.

By (111), (112) and (9) and (20), the web (109) belongs to the class $\mathbf{A_{21}}$. It is interesting to find out for which values of c^2_{jk} the web (109) belongs to the classes $\mathbf{C}$ or $\mathbf{D}$ and to the class $\mathbf{A_{22}}$.

Example 10 Take another particular case of the web (105) by taking $c^1_{11} = c^1_{21} = c^1_{22} = c^2_{11} = c^2_{12} = c^2_{22} = 0$ and $c^1_{12} = c^2_{21} = 1$:

$$u^1_3 = x^1 + y^1 + x^1 y^2, \quad u^2_3 = x^2 + y^2 + x^2 y^1 \qquad (114)$$

in a domain

$$\Delta_1 = (1 + y^1)(1 + y^2) \neq 0, \quad \Delta_2 = 1 - x^1 x^2 \neq 0.$$

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i.e., $y^i \neq -1$, $x^1 x^2 \neq 1$. In this case using (48)–(51) and (32)–(37), we find that

$$
\begin{cases}
\Gamma^1_{11} = -x^2 \Gamma^1_{12} = \dfrac{x^2 y^2}{\Delta_1 \Delta_{2_1}}, & \Gamma^2_{1i} = \Gamma^1_{2i} = 0, \\[2mm]
\Gamma^2_{22} = -x^1 \Gamma^2_{21} = \dfrac{x^1}{(1+y^1)\Delta_2},
\end{cases}
\tag{115}
$$

$$
\begin{cases}
\omega^1_1 = a_2 x^2 (\underset{1}{\omega^1} + \underset{2}{\omega^1}) - a_2 \underset{2}{\omega^2}, & \omega^2_1 = -a_1 \underset{1}{\omega^2}, \\[2mm]
\omega^2_2 = a_1 x^1 (\underset{1}{\omega^2} + \underset{2}{\omega^2}) - a_1 \underset{2}{\omega^1}, & \omega^1_2 = -a_2 \underset{1}{\omega^1},
\end{cases}
\tag{116}
$$

$$
a_1 = \frac{1}{(1+y^1)\Delta_2}, \quad a_2 = \frac{y^2}{\Delta_1 \Delta_2},
\tag{117}
$$

$$
\begin{cases}
p_{12} = q_{12} = a_1(a_1 x^1 + a_2), & q_{11} = -a_1(a_2 x^2 + a_1), \\[2mm]
p_{21} = q_{21} = a_2(a_2 x^2 + a_1), & q_{22} = -a_2(a_1 x^1 + a_2), \\[2mm]
p_{11} = p_{22} = 0.
\end{cases}
\tag{118}
$$

$$
p = q = \frac{a_1^2 x^1 - a_2^2 x^2}{2},
\tag{119}
$$

$$
\begin{cases}
b^i_{jjj} = b^1_{12i} = b^1_{2i1} = b^2_{i12} = b^2_{2i1} = 0, \\[2mm]
b^1_{112} = -\dfrac{x^2}{2(1+y^2)^2 \Delta_2^2}, & b^1_{212} = -\dfrac{x^1 + y^1 + x^1 y^2 + 1}{(1+y^2)\Delta_1 \Delta_2^2}, \\[2mm]
b^2_{121} = -\dfrac{x^2 + y^2 + x^2 y^1 + 1}{(1+y^1)\Delta_1 \Delta_2^2}, & b^2_{122} = \dfrac{x^1 + y^1 + x^1 y^2 + 1}{(1+y^1)\Delta_1 \Delta_2^2}.
\end{cases}
\tag{120}
$$

By (118) and (119), the web (114) belongs to the classes $\mathbf{E_{131}}$ and $\mathbf{G_3}$. Using (117) and (118), it is easy to check that condition (8) does not hold. Thus the web (114) belongs to the class $\mathbf{B}$.

Suppose that the web (114) is transversally geodesic. Then it follows from (38) and (120) that $b_{22} = \frac{3}{4} b^i_{(i22)} = \frac{1}{4} b^1_{212}$. As a result, by (38) and (120), we have $b^1_{(122)} = \frac{1}{3} b_{22} = \frac{1}{12} b^1_{212}$. On the other hand, $b^1_{(122)} = \frac{1}{3} b^1_{212}$. Since by (120), $b^1_{212} \neq 0$, we came to a contradiction. This contradiction proves that the web (114) is not transversally geodesic, i.e., this web belongs to the class $\mathbf{C}$.

Example 11 Consider the web defined by the equations

$$
u^1_3 = x^1 + y^1 + \frac{1}{2}(x^1)^2 y^2, \quad u^2_3 = x^2 + y^2 + \frac{1}{2} x^1 (y^2)^2
\tag{121}
$$

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in a domain of $\mathbf{R}^4$ where $\Delta = 1 + x^1 y^2 \neq 0$, i.e., $x^1 y^2 \neq -1$.

In this case using (48)–(51) and (32)–(37), we find that

$$\Gamma^1_{12} = -\frac{x^1}{\Delta^2}, \quad \Gamma^2_{12} = -\frac{y^2}{\Delta^2}, \quad \Gamma^1_{i1} = \Gamma^1_{22} = \Gamma^2_{i1} = \Gamma^2_{22} = 0, \tag{122}$$

$$\omega^1_1 = -\frac{x^1}{\Delta^2}\omega^2_2, \quad \omega^1_2 = -\frac{x^1}{\Delta^2}\omega^1_1, \quad \omega^2_1 = -\frac{y^2}{\Delta^2}\omega^2_2, \quad \omega^2_2 = -\frac{y^2}{\Delta^2}\omega^1_1, \tag{123}$$

$$a_1 = -\frac{y^2}{\Delta^2}, \quad a_2 = \frac{x^1}{\Delta^2}, \tag{124}$$

$$\begin{cases} p_{i2} = 0, & q_{i1} = 0, \\ p_{11} = \dfrac{2(y^2)^2}{\Delta^4}, & q_{22} = -\dfrac{2(x^1)^2}{\Delta^4}, \\ p_{21} = -q_{12} = \dfrac{1 - x^1 y^2}{\Delta^4}, \end{cases} \tag{125}$$

$$p = q = \frac{x^1 y^2 - 1}{2\Delta^4}, \tag{126}$$

$$\begin{cases} b^i_{jjj} = b^1_{12i} = b^1_{2i1} = b^2_{2i1} = b^2_{12i} = 0, \\ b^1_{112} = b^2_{112} = -b^2_{212} = \dfrac{x^1 y^2 - 1}{\Delta^4}, \quad b^1_{212} = -\dfrac{2(x^1)^2}{\Delta^3}. \end{cases} \tag{127}$$

By (125) and (126), the web (121) belongs to the classes $\mathbf{G_3}$ and $\mathbf{E_{111}}$. Recall that for webs of the class $\mathbf{E_{111}}$, both components a_1 and a_2 are covariantly constant on the 2-dimensional distribution defined by the equations $\omega^1_1 = \omega^2_2 = 0$, and they satisfy the exterior quadratic equation $\nabla a_1 \wedge \omega^1_1 = \nabla a_2 \wedge \omega^2_2$ which is equivalent to the condition $p_{21} = -q_{12}$. We establish now a geometric meaning of the latter condition for the web (121).

Consider the 3-dimensional distribution defined by the equation $a_1 \omega^1_1 - a_2 \omega^2_2 = 0$. Taking exterior derivative of this equation and applying (123) and (124), we find that

$$\nabla a_1 \wedge \omega^1_1 - \nabla a_2 \wedge \omega^2_2 = 0.$$

It follows that *for the web (121), the condition $p_{21} = -q_{12}$ is necessary and sufficient for the distribution $a_1 \omega^1_1 - a_2 \omega^2_2 = 0$ to be integrable.*

Note also that by (123) and (124), we have

$$d(\omega^1_1 + \omega^1_2) = 0, \quad d(\omega^2_1 + \omega^2_2) = 0.$$

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This means that *for the web* (121), *the* 3-*dimensional distributions defined by the equations* $\omega^1 + \omega^1 = 0$ *and* $\omega^2 + \omega^2 = 0$ *are integrable.*
$\quad\;\;\; 1 \quad 2 \qquad\qquad\qquad 1 \quad 2$

Using (124) and (125), it is easy to check that the left-hand side of condition (8) is $\dfrac{2x^1(x^2)^2 + x^1y^2(1 - x^1y^2)}{\Delta^8} \neq 0$, i.e., condition (8) does not hold. Thus the web (121) belongs to the class **B**.

Suppose that the web (121) is transversally geodesic. Then it follows from (38) and (127) that $b_{12} = \frac{1}{4}b^i_{(i22)} = 0$. As a result, by (38) and (127), we have $b^1_{(112)} = \frac{2}{3}b_{12} = 0$. On the other hand $b^1_{(112)} = \frac{1}{3}b^1_{112}$. Since by (127), $b^1_{112} \neq 0$, we come to a contradiction. This contradiction proves that the web (121) is not transversally geodesic, i.e., this web belongs to the class **C**.

Example 12 Consider the web defined by the equations

$$u^1_3 = (x^1 + y^1)(x^2 - y^2), \quad u^2_3 = (x^1 - y^1)(x^2 + y^2) \tag{128}$$

in a domain of $\mathbf{R}^4$ where $\Delta = x^1y^2 + x^2y^1 \neq 0$.

In this case using (48)–(51) and (32)–(37), we find that

$$
\begin{cases}
\Gamma^1_{11} = -\Gamma^2_{11} = \dfrac{u^2_3}{2\Delta^2}, \quad \Gamma^1_{22} = -\Gamma^2_{22} = -\dfrac{u^1_3}{2\Delta^2}, \\[2mm]
\Gamma^1_{12} = -\Gamma^1_{21} = -\Gamma^2_{12} = \Gamma^2_{21} = \dfrac{\rho}{2\Delta^2},
\end{cases}
\tag{129}
$$

$$
\begin{cases}
\omega^1_1 = -\omega^2_1 = \dfrac{1}{2\Delta^2}\left[u^2_3(\omega^1 + \omega^1) - \rho(\omega^2 - \omega^2)\right], \\[2mm]
\qquad\qquad\qquad\qquad\quad\;\; 1 \quad\; 2 \qquad\quad 1 \quad\; 2 \\[-2mm]
\omega^2_2 = -\omega^1_2 = \dfrac{1}{2\Delta^2}\left[\rho(\omega^1 - \omega^1) + u^1_3(\omega^1 + \omega^1)\right],
\end{cases}
\tag{130}
$$

$$a_1 = a_2 = -\dfrac{\rho}{2\Delta^2}, \tag{131}$$

$$
\begin{cases}
p_{11} = p_{21} = \dfrac{\beta(x^1 - y^1) - \alpha(x^2 + y^2)}{2\Delta^4}, \\[2mm]
p_{12} = p_{22} = \dfrac{-\beta(x^1 + y^1) + \alpha(x^2 - y^2)}{2\Delta^4}, \\[2mm]
q_{11} = q_{21} = \dfrac{\gamma(x^1 - y^1) + \delta(x^2 + y^2)}{2\Delta^4}, \\[2mm]
q_{12} = q_{22} = \dfrac{\gamma(x^1 + y^1) + \delta(x^2 - y^2)}{2\Delta^4},
\end{cases}
\tag{132}
$$

$$p = q = \dfrac{\rho\sigma}{\Delta^4}, \tag{133}$$

$$
\begin{cases}
b^2_{i11} = -b^1_{i11} = \dfrac{u^2_3\rho}{\Delta^4}, \quad b^1_{222} = -b^2_{222} = -b^i_{122} = \dfrac{u^1_3\rho}{\Delta^4}, \\[2mm]
b^1_{i12} = -b^1_{i21} = -b^2_{i12} = b^2_{i21} = \dfrac{\sigma^2 - 2[(x^1x^2)^2 + (y^1y^2)^2]}{\Delta^4},
\end{cases}
\tag{134}
$$

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where

$$
\begin{cases}
\alpha = (x^1)^2 y^2 - x^1 x^2 y^1 - 2(y^1)^2 y^2, & \beta = (x^2)^2 y^1 - x^1 x^2 y^2 - 2y^1 (y^2)^2, \\
\gamma = x^1 (y^2)^2 - x^2 y^1 y^2 - 2x^1 (x^2)^2, & \delta = x^2 (y^1)^2 - x^1 y^1 y^2 - 2(x^1)^2 x^2, \\
\rho = x^1 x^2 + y^1 y^2, & \sigma = x^1 y^2 - x^2 y^1.
\end{cases}
$$

By (133), the web (128) belongs to the class $\mathbf{G_3}$. Equations (131) and (132) show that conditions (11) and (22) hold. Thus, the web (128) belongs to the class $\mathbf{A_{131}}$. It is easy to check that the curvature tensor of this web defined by equations (134) satisfies both equations (27). As a result, the web (128) belongs to the class $\mathbf{A_{132}}$.

It follows from (134) that $b^1_{222} \neq 0$, and by Corollary 8a), this proves that the web (121) is not transversally geodesic, i.e., this web belongs to the class $\mathbf{C}$.

Example 13 Consider the web defined by the equations

$$
u^1_3 = x^2 y^2 e^{x^1 y^1}, \quad u^2_3 = x^2 + y^2 \tag{135}
$$

in a domain of $\mathbf{R}^4$ where $x^i \neq 0$, $y^i \neq 0$. In this case using (48)–(51) and (32)–(37), we find that

$$
\begin{cases}
\Gamma^1_{12} = \dfrac{1}{x^1 y^1 y^2}, \ \ \Gamma^1_{21} = \dfrac{1}{x^1 x^2 y^1}, \ \ \Gamma^1_{22} = -\dfrac{e^{x^1 y^1}}{x^1 y^1}, \\[2mm]
\Gamma^1_{11} = -\dfrac{(1 + x^1 y^1) e^{-x^1 y^1}}{x^1 x^2 y^1 y^2}, \ \ \Gamma^2_{jk} = 0,
\end{cases} \tag{136}
$$

$$
\begin{cases}
\omega^1_1 = \dfrac{1}{x^1 x^2 y^1 y^2} \left[-(1 + x^1 y^1) e^{-x^1 y^1} (\underset{1}{\omega}^1 + \underset{2}{\omega}^1) + y^2 \underset{1}{\omega}^2 + x^2 \underset{2}{\omega}^2 \right], \\[2mm]
\omega^1_2 = \dfrac{1}{x^1 x^2 y^1 y^2} \left[x^2 \underset{1}{\omega}^1 + y^2 \underset{2}{\omega}^1 - x^2 y^2 e^{x^1 y^1} (\underset{1}{\omega}^2 + \underset{2}{\omega}^2) \right], \ \ \omega^2_i = 0,
\end{cases} \tag{137}
$$

$$
a_1 = 0, \quad a_2 = \frac{x^2 - y^2}{x^1 x^2 y^1 y^2}, \tag{138}
$$

$$
\begin{cases}
p_{1i} = q_{1i} = 0, & p_{21} = q_{21} = \dfrac{(y^2 - x^2) e^{-x^1 y^1}}{(x^1 x^2 y^1 y^2)^2}, \\[2mm]
p_{22} = \dfrac{x^2 - y^2 + x^1 y^1 y^2}{(x^1 x^2 y^1)^2 y^2}, & q_{22} = \dfrac{x^2 - y^2 - x^1 x^2 y^1}{(x^1 y^1 y^2)^2 x^2},
\end{cases} \tag{139}
$$

$$
p = q = \frac{(x^2 - y^2) e^{-x^1 y^1}}{2(x^1 x^2 y^1 y^2)^2}. \tag{140}
$$

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$$\begin{cases} b^2_{jkl} = b^1_{11i} = b^1_{12i} = 0, \\ b^1_{211} = \dfrac{(x^2 - y^2)(2 - x^1 y^1)e^{-x^1 y^1}}{(x^1 x^2 y^1 y^2)^2}, \quad b^1_{212} = \dfrac{2(y^2 - x^2) + x^1 x^2 y^1}{(x^1 y^1 y^2)^2 x^2}, \\ b^1_{222} = \dfrac{(x^2 - y^2)(2 - x^1 y^1)e^{x^1 y^1}}{(x^1 y^1)^2 x^2 y^2}, \quad b^1_{221} = \dfrac{2(y^2 - x^2) + x^1 y^1 y^2}{(x^1 x^2 y^1)^2 y^2}. \end{cases} \qquad (141)$$

By (139) and (140), the web (135) belongs to the classes $\mathbf{E_{32}}$ and $\mathbf{G_3}$. Equations (138), (139), and (141) show that conditions (10), (21), and (26) hold. Thus, the web (135) belongs to the classes $\mathbf{A_{31}}$ and $\mathbf{A_{32}}$.

It follows from (141) that $b^1_{222} \neq 0$, and as a result, according to Corollary 8a), the web (135) is not transversally geodesic, i.e., this web belongs to the class $\mathbf{C}$.

Example 14 Consider the web defined by the equations

$$u^1_3 = y^2 e^{x^1 y^1}, \quad u^2_3 = x^2 + y^2 \qquad (142)$$

in a domain of $\mathbf{R}^4$ where $x^1 \neq 0$, $y^i \neq 0$ (see [AS 92], Ch. 3, Problem **5**, p. 133). In this case using (48)–(51) and (32)–(37), we find that

$$\begin{cases} \Gamma^1_{11} = -\dfrac{(1 + x^1 y^1)e^{-x^1 y^1}}{x^1 y^1 y^2}, \quad \Gamma^1_{12} = \dfrac{1}{x^1 y^1 y^2}, \\ \Gamma^1_{2i} = 0, \quad \Gamma^2_{jk} = 0, \end{cases} \qquad (143)$$

$$\begin{cases} \omega^1_1 = \dfrac{1}{x^1 y^1 y^2}\left[-(1 + x^1 y^1)e^{-x^1 y^1}(\underset{1}{\omega^1} + \underset{2}{\omega^1}) + \underset{2}{\omega^2}\right], \\ \omega^1_2 = \dfrac{1}{x^1 y^1 y^2}\underset{1}{\omega^1}, \quad \omega^2_i = 0, \end{cases} \qquad (144)$$

$$a_1 = 0, \quad a_2 = -\dfrac{1}{x^1 y^1 y^2}, \qquad (145)$$

$$\begin{cases} p_{1i} = q_{1i} = 0, \quad p_{21} = q_{21} = \dfrac{e^{-x^1 y^1}}{(x^1 y^1 y^2)^2}, \\ p_{22} = 0, \quad q_{22} = -\dfrac{1}{(x^1 y^1 y^2)^2}, \end{cases} \qquad (146)$$

$$p = q = -\dfrac{e^{-x^1 y^1}}{2(x^1 y^1 y^2)^2}. \qquad (147)$$

$$\begin{cases} b^2_{jkl} = b^1_{11i} = b^1_{12i} = b^1_{22i} = 0, \\ b^1_{211} = \dfrac{e^{-x^1 y^1}(3 + 2x^1 y^1)}{(x^1 y^1 y^2)^2}, \quad b^1_{212} = \dfrac{x^1 y^1 - 1}{(x^1 y^1 y^2)^2}. \end{cases} \qquad (148)$$

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By (146) and (147), the web (142) belongs to the classes $\mathbf{E_{321}}$ and $\mathbf{G_3}$. Equations (145), (146), and (148) show that conditions (10), (21), and (26) hold. Thus, the web (142) belongs to the classes $\mathbf{A_{31}}$ and $\mathbf{A_{32}}$.

It follows from (148) that $b^2_{jkl} = 0$, and as a result, according to Corollary 8b), the web (142) is not transversally geodesic, i.e., this web belongs to the class $\mathbf{C}$.

Note that in [AS 92], Ch. 3, Problem **5**, p. 133, it is wrongly indicated that the web (142) is transversally geodesic.

Example 15 Consider the web defined by the equations

$$u^1_3 = e^{x^1 y^1} + x^2 y^2, \quad u^2_3 = x^2 + y^2 \tag{149}$$

in a domain of $\mathbf{R}^4$ where $x^1, y^2 \neq 0$ (see [AS 92], Ch. 3, Problem **8**, p. 133). In this case using (48)–(51) and (32)–(37), we find that

$$\begin{cases} \Gamma^1_{11} = -Ae^{-x^1 y^1}, \ \ \Gamma^1_{12} = Ax^2 e^{-x^1 y^1}, \ \ \Gamma^1_{21} = Ay^2 e^{-x^1 y^1}, \\ \Gamma^1_{22} = -Ax^2 y^2 e^{-x^1 y^1} - 1, \ \ \Gamma^2_{jk} = 0, \end{cases} \tag{150}$$

$$\begin{cases} \omega^1_1 = Ae^{-x^1 y^1}[-(\underset{1}{\omega^1} + \underset{2}{\omega^1}) + y^2 \underset{1}{\omega^2} + x^2 \underset{2}{\omega^2}], \ \ \omega^2_i = 0, \\ \omega^1_2 = Ae^{-x^1 y^1}[(x^2 y^2 + A^{-1} e^{x^1 y^1})[-(\underset{1}{\omega^2} + \underset{2}{\omega^2}) + x^2 \underset{1}{\omega^1} + y^2 \underset{2}{\omega^1}], \end{cases} \tag{151}$$

$$a_1 = 0, \ \ a_2 = A(x^2 - y^2)e^{-x^1 y^1}, \tag{152}$$

$$\begin{cases} p_{1i} = q_{1i} = 0, \ \ p_{22} = -y^2 p_{21} + Ae^{-x^1 y^1}, \ \ q_{22} = -x^2 p_{21}, \\ p_{21} = q_{21} = (y^2 - x^2)Be^{-2x^1 y^1}, \end{cases} \tag{153}$$

$$p = q = \frac{(x^2 - y^2)Be^{-2x^1 y^1}}{2}, \tag{154}$$

$$\begin{cases} b^2_{jkl} = b^1_{11i} = b^1_{12i} = 0, \\ b^1_{211} = (x^2 - y^2)(B + A^2)e^{-2x^1 y^1} \\ b^1_{212} = Ae^{-x^1 y^1} - B(x^2)^2 e^{-2x^1 y^1}, \\ b^1_{221} = -Ae^{-x^1 y^1} - (x^2 - y^2)y^2(B + A^2)e^{-2x^1 y^1}, \\ b^1_{222} = \frac{1}{2}(4x^2 - 3y^2)(Ae^{-x^1 y^1} + x^2 y^2 Be^{-2x^1 y^1}), \end{cases} \tag{155}$$

where

$$A = 1 + \frac{1}{x^1 y^1}, \ \ B = A + \frac{1}{(x^1 y^1)^2}.$$

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By (153) and (154), the web (149) belongs to the classes $\mathbf{E_{32}}$ and $\mathbf{G_3}$. Equations (152), (153), and (155) show that conditions (10), (21), and (26) hold. Thus, the web (142) belongs to the classes $\mathbf{A_{31}}$ and $\mathbf{A_{32}}$.

It follows from (155) that $b^1_{222} \neq 0$, and as a result according to Corollary 8a), the web (142) is not transversally geodesic, i.e., this web belongs to the class $\mathbf{C}$.

5. Examples of nonextendable nonisoclinic webs $W(3,2,2)$ with $p \neq 0$, $q \neq 0$, $p \neq q$, and condition (43) not held (Class $\mathbf{G_4}$).

Example 16 A web $W(3,2,r)$ is given by

$$u^1_3 = \frac{1}{6}(x^1 + y^1)^3 + \frac{1}{2}\Big[(x^1)^2 + (y^1)^2 + 2x^2 y^2\Big], \quad u^2_3 = x^2 + y^2 \tag{156}$$

in a domain where

$$\Delta_1 = \frac{1}{2}\Big(x^1 + y^1\Big)^2 + x^1 \neq 0, \quad \Delta_2 = \frac{1}{2}\Big(x^1 + y^1\Big)^2 + y^1 \neq 0$$

(see [G 88], Ch. 8, Example **8.1.29**, p. 392 and [G 92]).

In this case using (48)–(51) and (32)–(37), we find that

$$\begin{cases} \Gamma^1_{11} = -\dfrac{\alpha}{\Delta_1 \Delta_2}, \quad \Gamma^1_{12} = \dfrac{\alpha x^2}{\Delta_1 \Delta_2}, \\[2mm] \Gamma^1_{21} = \dfrac{\alpha y^2}{\Delta_1 \Delta_2}, \quad \Gamma^1_{22} = -\Big[\dfrac{\alpha x^2 y^2}{\Delta_1 \Delta_2} + 1\Big], \quad \Gamma^2_{jk} = 0, \end{cases} \tag{157}$$

$$\begin{cases} \omega^1_1 = \dfrac{\alpha}{\Delta_1 \Delta_2}\Big[-(\underset{1}{\omega}^1 + \underset{2}{\omega}^1) + y^2 \underset{1}{\omega}^2 + x^2 \underset{2}{\omega}^2\Big], \quad \omega^2_i = 0, \\[2mm] \omega^1_2 = \dfrac{\alpha}{\Delta_1 \Delta_2}\Big[-\Big(x^2 y^2 + \dfrac{\Delta_1 \Delta_2}{\alpha}\Big)(\underset{1}{\omega}^2 + \underset{2}{\omega}^2) + x^2 \underset{1}{\omega}^1 + y^2 \underset{2}{\omega}^1\Big], \end{cases} \tag{158}$$

$$a_1 = 0, \quad a_2 = -\frac{\alpha\beta}{\Delta_1 \Delta_2}, \tag{159}$$

$$\begin{cases} p_{1i} = 0, \quad p_{21} = \dfrac{\beta A}{\Delta_1^3 \Delta_2^2}, \quad p_{22} = -y^2 p_{21} - \dfrac{\alpha}{\Delta_1 \Delta_2}, \\[2mm] q_{1i} = 0, \quad q_{21} = \dfrac{\beta B}{\Delta_1^2 \Delta_2^3}, \quad q_{22} = -x^2 q_{21} + \dfrac{\alpha}{\Delta_1 \Delta_2}, \end{cases} \tag{160}$$

$$p = -\frac{\beta A}{2\Delta_1^3 \Delta_2^2}, \quad q = -\frac{\beta B}{2\Delta_1^2 \Delta_2^3}, \tag{161}$$

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$$\begin{cases}
b^1_{111} = \dfrac{x^1 - y^1}{\Delta^3_1 \Delta^3_2}\left(-\dfrac{3}{4}\alpha^4 - \dfrac{3}{2}\alpha^3 - \alpha^2 + x^1 y^1\right), \\[2mm]
b^1_{112} = \dfrac{(x^1 - y^1)x^2}{\Delta^3_1 \Delta^3_2}\left(\dfrac{3}{4}\alpha^4 - \dfrac{1}{2}\alpha^3 - \alpha^2 - x^1 y^1\right), \\[2mm]
b^1_{121} = b^1_{211} = \dfrac{1}{\Delta^3_1 \Delta^3_2}\Big[(y^2\Delta_2 - x^2\Delta_1)(\Delta_1\Delta_2 - \alpha^2(\Delta_1 + \Delta_2)) \\[2mm]
\qquad\qquad\qquad + \alpha(x^2\Delta^2_1 - y^2\Delta^2_2)\Big], \\[2mm]
b^1_{122} = \dfrac{\alpha(\alpha + 1)x^2 y^2}{\Delta^3_1 \Delta^3_2}\Big[\dfrac{1}{4}\alpha^4 + \dfrac{1}{2}\alpha^3 + \alpha(\alpha + 1)(x^1 - y^1) + x^1 y^1\Big], \\[2mm]
b^1_{212} = \dfrac{1}{\Delta^3_1 \Delta^3_2}\Big[\alpha^2(\alpha + 1)(\Delta_2 y^2 \beta - \Delta_1(x^2)^2) + y^2\beta\Delta^2_2(2\alpha - \dfrac{1}{2}\Delta_1) \\[2mm]
\qquad\qquad\qquad + (x^2)^2\Delta^2_1(1 - \alpha)\Big], \\[2mm]
b^1_{221} = \dfrac{\alpha}{4\Delta^3_1 \Delta^3_2}\Big\{y^2(x^2\Delta_1 - y^2\Delta_2)\Big[\dfrac{4x^1 y^1}{(x^2\Delta_1 - y^2\Delta_2)\alpha} - \alpha^2(3\alpha + 2)\Big] \\[2mm]
\qquad\qquad\qquad + 4x^2(y^2\Delta^2_2 - x^2\Delta^2_1) - 4\Delta^2_1\Delta^2_2\Big\}, \\[2mm]
b^1_{222} = \dfrac{\alpha}{\Delta^3_1 \Delta^3_2}\Big[-\beta\Delta^2_1\Delta^2_2 + x^2 y^2(\alpha + 1)(\Delta_1\Delta_2 - \alpha^2(x^2\Delta_1 + y^2\Delta_2)) \\[2mm]
\qquad\qquad\qquad - x^2 y^2(x^2\Delta^2_1 + y^2\Delta^2_2)\Big], \quad b^2_{jkl} = 0.
\end{cases} \tag{162}$$

where

$$\begin{cases}
\alpha = x^1 + y^1, & \beta = x^2 - y^2, \\[2mm]
A = \dfrac{3}{4}\alpha^4 + \alpha^3 + (y^1)^2, & B = \dfrac{3}{4}\alpha^4 + \alpha^3 + (x^1)^2.
\end{cases}$$

By (160), the web (156) belongs to the class $\mathbf{E_3}$. Equations (159), (160), and (162) show that conditions (10) and (21) hold but condition (26) does not hold. Thus, the web (156) belongs to the class $\mathbf{A_{31}}$ and does not belong to the class $\mathbf{A_{32}}$.

It follows from (162) that $b^1_{222} \neq 0$, and as a result, according to Corollary 8a), the web (156) is not transversally geodesic, i.e., this web belongs to the class $\mathbf{C}$.

We now prove that *the web* (156) *cannot be expanded to a web* $W(4, 2, 2)$ *of maximum 2-rank*. Thus we have to prove that the relations (43) do not hold for the web (149).

It follows from (36) and (151) that

$$dp = p\omega^1_1 + \underset{1}{p}_i\underset{1}{\omega}^i + \underset{2}{p}_i\underset{2}{\omega}^i, \quad dq = q\omega^1_1 + \underset{1}{q}_i\underset{1}{\omega}^i + \underset{2}{q}_i\underset{2}{\omega}^i,$$

where ω^1_1 is defined by (158). On the other hand, we can find dp and dq by

differentiating their expressions (161). Comparing the results, we find that

$$
\begin{cases}
\underset{1}{p_1} = -\dfrac{3\alpha^2(\alpha+1)\beta}{2\Delta_1^4\Delta_2^2} + \dfrac{3A(\alpha+1)\beta}{2\Delta_1^5\Delta_2^2} + \dfrac{A\alpha\beta}{2\Delta_1^4\Delta_2^3}, & \underset{1}{p_2} = -y^2\underset{1}{p_1} - \dfrac{A}{2\Delta_1^3\Delta_2^2}, \\[4mm]
\underset{2}{p_1} = -\dfrac{[3\alpha^2(\alpha+1)+2y^1]\beta}{2\Delta_1^3\Delta_2^3} + \dfrac{A\alpha\beta}{\Delta_1^4\Delta_2^3} + \dfrac{A(\alpha+1)\beta}{\Delta_1^3\Delta_2^4}, & \underset{2}{p_2} = -x^2\underset{2}{p_1} + \dfrac{A}{2\Delta_1^3\Delta_2^2}, \\[4mm]
\underset{1}{q_1} = -\dfrac{[3\alpha^2(\alpha+1)+2x^1]\beta}{2\Delta_1^3\Delta_2^3} + \dfrac{B\alpha\beta}{\Delta_1^3\Delta_2^4} + \dfrac{B(\alpha+1)\beta}{\Delta_1^4\Delta_2^3}, & \underset{1}{q_2} = -y^2\underset{1}{q_1} - \dfrac{B}{2\Delta_1^2\Delta_2^3}, \\[4mm]
\underset{2}{q_1} = -\dfrac{3\alpha^2(\alpha+1)\beta}{2\Delta_1^2\Delta_2^4} + \dfrac{3B(\alpha+1)\beta}{2\Delta_1^2\Delta_2^5} + \dfrac{B\alpha\beta}{2\Delta_1^3\Delta_2^4}, & \underset{2}{q_2} = -x^2\underset{2}{q_1} + \dfrac{B}{2\Delta_1^2\Delta_2^3}.
\end{cases}
\tag{163}
$$

Let us assume that equation (43) holds for $i = 1$. Since by (159), $a_1 = 0$, this means that

$$
q(q\underset{1}{p_1} - p\underset{1}{q_1}) - p(q\underset{2}{p_1} - p\underset{2}{q_1}) = 0.
\tag{164}
$$

By (163) and (164), the left-hand side LHS of equation (43) taken for $i = 2$ is

$$
\text{LHS} = q(q\underset{1}{p_2} - p\underset{1}{q_2}) - p(q\underset{2}{p_2} - p\underset{2}{q_2}) = \beta p(q\underset{2}{p_1} - p\underset{2}{q_1}).
$$

Applying (163), from the last equation we find that

$$
\text{LHS} = \frac{\beta^3 p}{4\Delta_1^6\Delta_2^7}\Bigg\{ AB\Big[(\alpha+1)x^1 - \alpha y^1 + \frac{1}{2}\alpha^2\Big] \\
+ \Delta_1\Delta_2\Big[3\alpha^3(\alpha+1)(x^1 - y^1) + 2By^1\Big]\Bigg\}.
\tag{165}
$$

On the other hand, by (159) and (161), for $i = 2$ the right-hand side RHS of (43) is

$$
\text{RHS} = \frac{\beta^3 p}{4\Delta_1^6\Delta_2^7}\Bigg\{ AB(-A\Delta_2 + B\Delta_1) \Bigg\}.
\tag{166}
$$

It is easy to see that in the curl brackets of expression (165) the highest degree of x^1 is 11 while in those of (166) the highest degree of x^1 is 14. This proves that the 2nd equation of (43) fails. Thus, by Theorem 7, part e), the web (149) cannot be expanded to a web $W(4, 2, 2)$ of maximum 2-rank. As a result, this web belongs to the class $\mathbf{G_4}$.

Example 17 Consider the web defined by

$$
u_3^1 = x^1 + y^1 + \frac{1}{2}(x^1)^2 y^2, \quad u_3^2 = x^2 + y^2 + \frac{1}{2}(x^2)^2 y^1
\tag{167}
$$

in a domain of $\mathbf{R}^4$ where $\Delta_1 = (1 + x^1y^2)(1 + x^2y^1) \neq 0$ and $\Delta_2 = \frac{1}{4}\Big[4 - (x^1x^2)^2\Big] \neq 0$, i.e., $x^1y^2 \neq -1$, $x^2y^1 \neq -1$, $x^1x^2 \neq \pm 2$.

In this case using (48)–(51) and (32)–(37), we find that

$$
\begin{cases}
\Gamma^1_{11} = \dfrac{\beta x^1 (x^2)^2}{2\Delta_1\Delta_2}, & \Gamma^2_{22} = \dfrac{\alpha(x^1)^2 x^2}{2\Delta_1\Delta_2}, \\[2mm]
\Gamma^1_{12} = -\dfrac{\beta x^1}{\Delta_1\Delta_2}, & \Gamma^2_{21} = -\dfrac{\alpha x^2}{\Delta_1\Delta_2}, \\[2mm]
\Gamma^1_{2i} == 0, & \Gamma^2_{1i} = 0,
\end{cases}
\tag{168}
$$

$$
\begin{cases}
\omega^1_1 = \dfrac{\beta x^1 (x^2)^2}{2\Delta_1\Delta_2}(\underset{1}{\omega^1} + \underset{2}{\omega^1}) - \dfrac{\beta x^1}{\Delta_1\Delta_2}\underset{2}{\omega^2}, & \omega^1_2 = -\dfrac{\beta x^1}{\Delta_1\Delta_2}\underset{1}{\omega^1}, \\[2mm]
\omega^2_2 = \dfrac{\alpha(x^1)^2 x^2}{2\Delta_1\Delta_2}(\underset{1}{\omega^2} + \underset{2}{\omega^2}) - \dfrac{\alpha x^2}{\Delta_1\Delta_2}\underset{2}{\omega^1}, & \omega^2_1 = -\dfrac{\alpha x^2}{\Delta_1\Delta_2}\underset{1}{\omega^2},
\end{cases}
\tag{169}
$$

$$
a_1 = \frac{\alpha x^2}{\Delta_1\Delta_2}, \quad a_2 = \frac{\beta x^1}{\Delta_1\Delta_2},
\tag{170}
$$

$$
\begin{cases}
p_{11} = 0, & q_{11} = -\dfrac{A(x^2)^2}{2\Delta_1\Delta_2^2\beta}, \\[2mm]
p_{12} = \dfrac{1}{\Delta_2\beta^3} + \dfrac{(x^1x^2)^2}{2\Delta_2^2\beta^2} + \dfrac{x^1x^2}{\Delta_1\Delta_2}, & q_{12} = \dfrac{Bx^1x^2}{2\Delta_1\Delta_2^2\beta}, \\[2mm]
p_{21} = \dfrac{1}{\Delta_2\alpha^3} + \dfrac{(x^1x^2)^2}{2\Delta_2^2\alpha^2} + \dfrac{x^1x^2}{\Delta_1\Delta_2}, & q_{21} = \dfrac{Ax^1x^2}{2\Delta_1\Delta_2^2\alpha}, \\[2mm]
p_{22} = 0, & q_{22} = -\dfrac{B(x^1)^2}{2\Delta_1\Delta_2^2\alpha},
\end{cases}
\tag{171}
$$

$$
p = q + \frac{1}{2\Delta_2}\left(\frac{1}{\beta^3} - \frac{1}{\alpha^3}\right), \quad q = \frac{(x^1x^2)^2}{4\Delta_2^2}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right),
\tag{172}
$$

$$
\begin{cases}
b^1_{111} = \dfrac{(x^2)^2}{2\alpha^3\Delta_2^2}\Big[\Delta_2 + \dfrac{\alpha(x^1x^2)^2}{4}\Big], & b^1_{121} = -\dfrac{x^1x^2}{4\Delta_1\Delta_2^2}, \\[2mm]
b^1_{112} = \dfrac{1}{4\alpha^3\Delta_2^2}\Big[-2\Delta_2 + \alpha(x^1x^2)^2\Big], & b^1_{211} = \dfrac{x^1x^2}{8\alpha\Delta_1\Delta_2^2}\Big[4\alpha - \beta x^1(x^2)^3\Big], \\[2mm]
b^1_{212} = -\dfrac{(x^1)^2}{4\alpha\Delta_2^2}\Big[\alpha x^1 x^2(1 + x^2) + 2\beta(2 + (x^2)^2)\Big], & b^1_{i22} = b^1_{221} = 0, \\[2mm]
b^2_{222} = \dfrac{(x^1)^2}{8\beta^3\Delta_2^2}\Big[4\Delta_2 + \beta(x^1x^2)^2\Big], & b^2_{121} = -\dfrac{A(x^2)^2}{2\beta\Delta_1\Delta_2^2}, \\[2mm]
b^2_{122} = \dfrac{x^1x^2}{2\beta\Delta_1\Delta_2}\Big[2\alpha x^1 x^2 + \beta(1 - x^2)\Big], & b^2_{212} = \dfrac{x^1(x^2)^2}{2\Delta_1\Delta_2^2}, \\[2mm]
b^2_{221} = -\dfrac{(x^1)^2}{4\beta^3\Delta_2^2}\Big[4\Delta_2 + \beta(x^1x^2)^2\Big], & b^2_{i11} = b^2_{112} = 0,
\end{cases}
\tag{173}
$$

32

where

$$\begin{cases} \alpha = 1 + x^1 y^2, & \beta = 1 + x^2 y^1, \\ A = 2\alpha + \beta x^1 x^2, & B = 2\beta + \alpha x^1 x^2. \end{cases}$$

It follows from equations (170) and (171) that conditions (8) do not hold. In fact, by (171), the first and the third terms of the left-hand side of the first equation of (8) vanish. Up to a common factor, the middle term of the first equation of (8) is

$$2\alpha\beta\Delta_2(\alpha^3 + \beta^3) + (x^1)^2(x^2)^2\alpha^2\beta^2(\alpha^2 + \beta^2) + 2x^1 x^2\Big(1 - \frac{(x^1)^2(x^2)^2}{4}\Big).$$

The degree of the 2nd term of this expression (with respect to x^i and y^j) is 16, and it is higher than the degrees 14 and 6 of two other terms. This term is

$$(x^1)^2(x^2)^2(1 + x^1 y^2)^2(1 + x^2 y^1)^2(2 + (x^1)^2(y^2)^2 + (x^2 y^1)^2 + 2x^1 y^2 + 2x^2 y^1).$$

The highest degree terms of the last expression are

$$(x^1)^4(x^2)^4(y^1)^2(y^2)^2((x^1)^2(y^2)^2 + (x^2 y^1)^2)$$

do not vanish and have no similar terms with two other terms of the left-hand side of the first equation of (8). Thus the web (167) belongs to the class **B**.

Equations (171) prove that the web (167) belongs to the class $\mathbf{E_{13}}$.

It follows from (36) that

$$dp = p(\omega_1^1 + \omega_2^2) + \underset{1}{p_i}\omega_1^i + \underset{2}{p_i}\omega_2^i, \quad dq = q(\omega_1^1 + \omega_2^2) + \underset{1}{q_i}\omega_1^i + \underset{2}{q_i}\omega_2^i,$$

where ω_1^1 and ω_2^2 are defined by (169). On the other hand, we can find dp and dq by differentiating their expressions (172). Comparing the results, we find that

$$\begin{cases} \underset{1}{q_1} = \dfrac{x^1(x^2)^2(4 + (x^1 x^2)^2)}{4\alpha\Delta_2^3}\Big(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\Big) + \dfrac{(x^1 x^2)^2 y^2}{\alpha^4\Delta_2^2}, \\ \underset{1}{q_2} = \dfrac{(x^1)^2 x^2(4 + (x^1 x^2)^2)}{4\beta\Delta_2^3}\Big(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\Big) - \dfrac{(x^1 x^2)^2 y^1}{\beta^4\Delta_2^2}, \\ \underset{2}{q_1} = \dfrac{(x^1)^2(x^2)^3}{4\alpha\Delta_2^2}\Big[\dfrac{3x^1 y^1}{\alpha^2\Delta_2} - \dfrac{1}{\beta}\Big(\dfrac{x^1 x^2}{\beta} + \dfrac{2}{\alpha}\Big)\Big], \ \underset{1}{p_1} = \underset{1}{q_1} + \dfrac{3y^2}{2\alpha^5\Delta_2}, \\ \underset{2}{q_2} = \dfrac{(x^1)^3(x^2)^2}{4\beta\Delta_1^2\Delta_2^3}\Big[x^1 y^1\Big(\alpha^2 + \beta^2\Big) + 2\Delta_1\Big], \ \underset{1}{p_2} = \underset{1}{q_2} - \dfrac{3y^1}{2\beta^5\Delta_2}, \\ \underset{2}{p_1} = \underset{2}{q_1} - 2K - (x^2)^2 L, \ \underset{2}{p_2} = \underset{2}{q_2} + (x^1)^2 K + 2L, \end{cases} \qquad (174)$$

where

$$K = \frac{x^2}{4\beta\Delta_2^2}\Big(\frac{1}{\alpha^3} + \frac{2}{\beta^3}\Big), \quad L = \frac{x^1}{4\alpha\Delta_2^2}\Big(\frac{2}{\alpha^3} + \frac{1}{\beta^3}\Big).$$

Consider the first of two equations (43). Substitute the values of p, q, a_1, and $\underset{\alpha}{p_1}, \underset{\alpha}{q_1}$ from (172), (170), and (174) into this equation, and collect all similar

terms. It turns out that the highest (53rd) degree term is $-48(x^1x^2)^{18}(y^1)^9(y^2)^8 \neq 0$. Thus, the web (167) belongs to the class $\mathbf{G_4}$.

Finally, we prove that the web (167) is not transversally geodesic. Suppose that it is transversally geodesic. By (173), the component b_{112}^1 of the curvature tensor of the web (167) vanishes, $b_{112}^1 = 0$. Since this web is transversally geodesic, by (38), we have $b_{112}^1 = \frac{1}{3}b_{11}$. But as we noted earlier, $b_{11} = \frac{3}{4}b_{(k11)}^k$. By (173), we find that $b_{11} = \frac{1}{4}(3b_{111}^1 + b_{121}^2)$. A straightforward calculation shows that b_{11} is proportional to a polynomial in x^i and y^j whose highest degree term is $S(x^1)^3(x^2)^4y^2(y^1)^2$ with $S \neq 0$. Thus $b_{11} \neq 0$, and consequently $b_{112}^1 \neq 0$. This contradiction proves that the web (167) belongs to the class $\mathbf{C}$.

Example 18 Consider the web defined by the equations

$$u_3^1 = x^1 + y^1 + x^1y^2, \quad u_3^2 = x^1y^1 + x^2y^2 \tag{175}$$

in a domain of $\mathbf{R}^4$ where $\Delta_1 = y^2(1 + y^2) \neq 0$, and $\Delta_2 = x^2 - (x^1)^2 \neq 0$, i.e., $y^2 \neq 0, -1$ and $(x^1)^2 \neq x^2$.

In this case using (48)–(51) and (32)–(37), we find that

$$
\begin{cases}
\Gamma_{11}^1 = \dfrac{x^1y^2}{\Delta_1\Delta_2}, & \Gamma_{12}^1 = -\dfrac{x^2y^2}{\Delta_1\Delta_2}, & \Gamma_{12}^2 = \dfrac{x^1y^2 + y^1}{\Delta_1\Delta_2}, \\[2mm]
\Gamma_{22}^2 = -\dfrac{1}{y^2\Delta_2}, & \Gamma_{11}^2 = -\dfrac{x^1y^1 + x^2y^2}{\Delta_1\Delta_2}, & \Gamma_{21}^2 = \dfrac{x^1}{y^2\Delta_2}, \quad \Gamma_{2i}^1 = 0,
\end{cases}
\tag{176}
$$

$$
\begin{cases}
\omega_1^1 = \dfrac{y^2}{\Delta_1\Delta_2}\left[x^1(\underset{1}{\omega^1} + \underset{2}{\omega^1}) - x^2\underset{2}{\omega^2}\right], \quad \omega_2^1 = -\dfrac{x^2y^2}{\Delta_1\Delta_2}\underset{1}{\omega^1}, \\[2mm]
\omega_1^2 = -\dfrac{1}{\Delta_1\Delta_2}\left[(x^1y^1 + x^2y^2)(\underset{1}{\omega^1} + \underset{2}{\omega^1}) + x^1(1 + y^2)\underset{1}{\omega^2} + (x^1y^2 + y^1)\underset{2}{\omega^2}\right], \\[2mm]
\omega_2^2 = \dfrac{1}{\Delta_1\Delta_2}\left[(x^1y^2 + y^1)\underset{1}{\omega^1} - (1 + y^2)(\underset{1}{\omega^2} + \underset{2}{\omega^2}) + x^1(1 + y^2)\underset{2}{\omega^1}\right],
\end{cases}
\tag{177}
$$

$$a_1 = \dfrac{y^1 - x^1}{\Delta_1\Delta_2}, \quad a_2 = \dfrac{x^2y^2}{\Delta_1\Delta_2}, \tag{178}$$

$$
\begin{cases}
p_{11} = \dfrac{1}{\Delta_1^2\Delta_2^2}\left[(x^1y^1 + x^2y^2)(1 - x^2y^2) - x^1y^2(2x^1 - 3y^1) + (y^1)^2\right], \\[2mm]
q_{11} = \dfrac{1}{\Delta_1^2\Delta_2^2}\left[(x^1y^1 + x^2y^2)(1 + x^2 + y^2) - (x^1)^2(1 + y^2)\right], \\[2mm]
p_{12} = \dfrac{1}{\Delta_1^2\Delta_2y^2}\left[x^1 - y^1 - x^1x^2y^2\right], \\[2mm]
p_{21} = \dfrac{1}{\Delta_1^2\Delta_2^2}\left[x^1y^2(x^2y^2 + x^1y^1 + x^1 - y^1) - x^2y^1y^2\right], \\[2mm]
q_{12} = \dfrac{1}{\Delta_1^2\Delta_2^2}\left[(x^1 + y^1)y^2 + (1 + y^2)(x^1 + y^1 - x^1y^2 - x^1x^2y^2)\right], \\[2mm]
q_{21} = -\dfrac{x^1x^2y^2}{\Delta_1^2\Delta_2^2}, \quad p_{22} = \dfrac{1}{\Delta_1\Delta_2}, \quad q_{22} = \dfrac{x^2y^2}{\Delta_1^2\Delta_2^2},
\end{cases}
\tag{179}
$$

$$\begin{cases} p = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[(x^1 - y^1)(1 + y^2) - x^1(x^2 + y^1)\Delta_1 + x^2 y^1 y^2)\Big], \\[2mm] q = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[x^1\Big(1 + y^2 - (y^2)^2(x^2 + 1)\Big) + y^1\Big(1 + 2y^2\Big)\Big], \end{cases} \tag{180}$$

$$\begin{cases} b_{111}^1 = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[(x^1 y^1 + x^2 y^2)(2y^2 + x^1(1 + y^2))\Big], \\[2mm] b_{111}^2 = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[y^1 y^2(x^1 - 3x^2 - 2(x^1)^2) + 2x^1 x^2 y^2(1 - y^2) - (x^2)^2 y^1\Big], \\[2mm] b_{112}^1 = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[x^2 y^2(x^1 y^1 + x^2 y^2 - 3x^1 y^2) + x^1 y^2(y^2 - 2x^1 y^1)\Big], \\[2mm] b_{121}^1 = \dfrac{x^1(x^2 - 1)}{2\Delta_1\Delta_2^2}, \quad b_{211}^1 = \dfrac{x^2(y^2)^2 - x^1\Delta_1}{2\Delta_1^2\Delta_2^2}, \quad b_{222}^2 = \dfrac{1 - y^2}{(y^2)^2\Delta_2^2}, \\[2mm] b_{122}^1 = b_{212}^1 = -\dfrac{1}{2\Delta_1\Delta_2}, \quad b_{22i}^1 = b_{221}^2 = 0, \\[2mm] b_{112}^2 = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[x^1 y^2(2x^2 y^2 - 2x^1 - x^2 - y^2 - 1 + x^2 y^1 - 3x^1 y^2) \\[1mm] \qquad\qquad + y^1 y^2(x^2 - 4x^1) - (x^1)^2 y^1 + x^2 y^2(x^2 y^2 - y^2 - 1)\Big], \\[2mm] b_{121}^2 = \dfrac{1}{2\Delta_1\Delta_2^2 y^2}\Big[x^1(2y^1 - 2x^1(1 + y^2) - y^2) + x^2(x^2 + y^2(3 - x^1))\Big], \\[2mm] b_{211}^2 = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[x^1 y^2(y^2(y^1 - x^1 - 2) - x^2 y^1 - 2) \\[1mm] \qquad\qquad + x^2(y^2(x^2 + y^2 - y^1) - (y^2)^2 + x^2 - y^1)\Big], \\[2mm] b_{122}^2 = \dfrac{1}{2\Delta_1\Delta_2^2 y^2}\Big[x^1(2 - y^1 - x^1 y^2 - x^2 y^2) - y^1(1 + y^2)\Big], \\[2mm] b_{212}^2 = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[y^2(x^1(x^2(1 + y^2) - 1 + y^2) + 2y^1)\Big], \end{cases} \tag{181}$$

Since $b_{111}^2 \neq 0$ (see (181)), it follows that the web (175) is nontransversally geodesic. So, it belongs to the class **C**.

It follows from (180) that

$$p - q = \dfrac{1}{2\Delta_1^2\Delta_2^2}\Big[x^2 y^1 y^2 - x^1 y^2(1 + y^2)(x^2 + y^1) + x^1(y^2)^2(1 + x^2) - y^1(3y^2 + 2)\Big],$$

i.e., $p \neq q$, and the inequalities (42) are satisfied.

It follows from (36) that

$$dp = p(\omega_1^1 + \omega_2^2) + \underset{1}{p_i}\,\underset{1}{\omega^i} + \underset{2}{p_i}\,\underset{2}{\omega^i}, \quad dq = q(\omega_1^1 + \omega_2^2) + \underset{1}{q_i}\,\underset{1}{\omega^i} + \underset{2}{q_i}\,\underset{2}{\omega^i},$$

where ω_1^1 and ω_2^2 are defined by (177). On the other hand, we can find dp and dq

by differentiating their expressions (180). Comparing the results, we find that

$$
\begin{cases}
\underset{1}{p_1} = \dfrac{1 - (x^2 + y^1)y^2 + x^1 y^1 - \dfrac{(y^1)^2 y^2}{\Delta_1}}{2\Delta_1^2 \Delta_2^2} + \dfrac{p(3y^1 + 5x^1 y^2)}{\Delta_1 \Delta_2}, \\[4mm]
\underset{1}{p_2} = \dfrac{y^1 - x^1(1 + y^2)}{2\Delta_1^2 \Delta_2^2} - \dfrac{3p}{y^2 \Delta_2}, \\[4mm]
\underset{1}{q_1} = \dfrac{1}{2\Delta_1^2 \Delta_2^3}\Big[x^1(-x^1 + y^1 + x^1(x^2 + y^1)(2y^2 + 1) - x^2(y^1 + y^2 + 1)) \\[2mm]
\qquad\qquad - x^2(x^2 y^2 + y^2 + 1)\Big] + \dfrac{3px^1(1 + 2y^2)}{\Delta_1 \Delta_2}, \\[4mm]
\underset{1}{q_2} = \dfrac{1}{2\Delta_1^2 \Delta_2^3}\Big[x^1\big(2 - (x^2 + y^1)(2y^2 + 1) + x^1 \Delta_1 + y^2(1 - x^2) \\[2mm]
\qquad\qquad - y^1(1 + x^2)\big)\Big] - \dfrac{p(3 + 5y^2)}{\Delta_1 \Delta_2}, \\[4mm]
\underset{2}{p_1} = \dfrac{y^2}{2\Delta_1^3 \Delta_2^2}\Big[1 + y^2(1 - y^2(x^2 + 1) + x^1 y^1 y^2)\Big] + \dfrac{3q(y^1 + 2x^1 y^2)}{\Delta_1 \Delta_2}, \\[4mm]
\underset{2}{p_2} = -\dfrac{x^1 y^2}{2\Delta_1^2 \Delta_2^2} - \dfrac{3q}{\Delta_1}, \\[4mm]
\underset{2}{q_1} = \dfrac{1}{2\Delta_1^2 \Delta_2^3}\Big[(x^1)^2(2y^2(x^2 + 1) - 1) + x^2(1 + 2y^2) - 2x^1 y^1\Big] \\[2mm]
\qquad\quad + \dfrac{3qx^1(1 + 2y^2)}{\Delta_1 \Delta_2}, \\[4mm]
\underset{2}{q_2} = \dfrac{1}{2\Delta_1^2 \Delta_2^3}\Big[x^1\big(1 - 2y^2(x^2 + 1) - 2y^2 - 1\big) + 2y^1\Big] + \dfrac{2q[1 + y^2(3 - x^2)]}{\Delta_1 \Delta_2}.
\end{cases}
\tag{182}
$$

If we substitute the values of a_i from (178), p, q from (180), and $\underset{\alpha}{p_1}$, $\underset{\alpha}{q_1}$, $\alpha = 1, 2$, from (182) into the first equation (43), multiply the result by the common denominator, and collect similar terms, we will see that there is the term with $12(x^1)^4 \neq 0$. Thus the web (175) belongs to the class $\mathbf{G_4}$.

It is easy to prove by means of equations (178) and (179) that conditions (8) do not hold. In fact, if we substitute the values of a_i from (178) and p_{ij} from (179) into the first equation of (8), we can observe that there is a term $(x^2 y^2)^4$ which do not have similar terms. Thus the web (175) belongs to the class $\mathbf{B}$.

We will present the results of this section in the following table in which we indicate to which classes the webs of our 18 examples belong.

Example/Class	A	B	C	D	E	F	G
1		B	C		E_{11}	F	
2	$A_{21} \cap A_{22}$		C		E_{22}		G_1
3		B	C		E_{12}		G_1
4	$A_{21} \cap A_{22}$		C		E_{22}		G_2
5	$A_{31} \cap A_{32}$		C		E_{31}		G_3
6	$A_{21} \cap A_{22}$		C		E_{23}		G_3
7	$A_{31} \cap A_{32}$		C		E_{32}		G_3
8			?	?			G_3
9	A_{21}		?	?	E_{23}		G_3
10		B	C		E_{131}		G_3
11		B	C		E_{111}		G_3
12	$A_{131} \cap A_{132}$		C				G_3
13	$A_{31} \cap A_{32}$		C		E_{32}		G_3
14	$A_{31} \cap A_{32}$		C		E_{321}		G_3
15	$A_{31} \cap A_{32}$		C		E_{32}		G_3
16	A_{31}		C		E_3		G_4
17		B	C		E_{13}		G_4
18		B	C				G_4

Thus the examples considered in this section prove the existence of all webs indicated in this table. Moreover, this proves the existence of more general webs than those indicated in the table. For example, the existence of A_{131} proves the existence of A_{13} and A_1.

Remark 15 We can see from the above table that the 3-webs of Examples **7** and **13** belong to the same classes. In order to show that they are not equivalent, we compare the vanishing components of their curvature tensor. In Example **7**, they are

$$b^2_{jkl} = b^1_{1kl} = b^1_{211} = 0.$$

In Example **13**, they are

$$b^2_{jkl} = b^1_{11i} = b^1_{11i} = 0.$$

Thus the 3-webs in these two examples are not equivalent.

We also can see from the above table that the 3-webs of Examples **13** and **15** belong to the same classes. It follows from (141) and (155) that they have the same vanishing components of their curvature tensors. So, some additional investigation is needed in order to determine whether the 3-webs of Examples **13** and **15** are equivalent.

Remark 16 In the table above for the webs of Examples 8 and 9, we put the question marks in the columns **C** and **D** since for certain values of the coefficients c^i_{jk} the polynomial webs of these examples belong to the class **C**, and for other values of these coefficients, they belong to the class **D**.

One can also see from the table above that in our examples there is no 3-webs that are transversally geodesic (Class **D**). This gives a rise to the following problem: *Construct an example of a nonisoclinic transversally geodesic nonhexagonal (or hexagonal) 3-web.*

References

[AG 96]   Akivis, M. A. and V. V. Goldberg, *Conformal differential geometry*, John Wiley & Sons, 1996, xiv+383 pp.

[AG 98]   Akivis, M. A. and V. V. Goldberg, *On four-dimensional three-webs with integrable transversal distributions*, Rend. Sem. Mat. Messina Ser. II, **5 (20)** (1998), 33–52 (2000).

[AG 00]   Akivis, M. A. and V. V. Goldberg, *Differential geometry of webs*, Chapter 1 in *Handbook of Differential Geometry*, Vol. 1, pp. 1–152, Elsevier Science B. V., 2000, 1–152.

[AS 71]   Akivis, M. A. and A. M. Shelekhov, *The computation of the curvature and torsion tensors of a multidimensional three-web and of the associator of the local quasigroup that is connected with it* (Russian) Sibirsk. Mat. Zh. **12** (1971), no. 5, 953–960. English translation: Siberian Math. J. **12** (1971), no. 5, 685–689.

[AS 92]   Akivis, M. A. and A. M. Shelekhov, *Geometry and algebra of nultidimensional three-Webs*, Translated from the Russian by V. V. Goldberg. Kluwer Academic Publishers, Dordrecht, 1992, xvii+358 pp.

[G 83]   Goldberg, V. V., *Tissus de codimension r et de r-rang maximum*, C. R. Acad. Sci. Paris Sér. I Math. **297** (1983), no. 6, 339–342.

[G 85]   Goldberg, V. V., *4-tissus isoclines exceptionnels de codimension deux et de 2-rang maximal*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 11, 593–596.

[G 86]   Goldberg, V. V., *Isoclinic webs $W(4,2,2)$ of maximum 2-rank*, Differential Geometry, Peniscola 1985, 168–183, Lecture Notes in Math., **1209**, Springer, Berlin-New York, 1986.

[G 87]   Goldberg, V. V., *Nonisoclinic 2-codimensional 4-webs of maximum 2-rank*, Proc. Amer. Math. Soc. **100** (1987), no. 4, 701–708.

[G 88]   Goldberg, V. V., *Theory of Multicodimensional $(n+1)$-Webs*, Kluwer Academic Publishers, Dordrecht, 1988, xxii + 466 pp.

[G 92]   Goldberg, V. V., *Rank problems for webs $W(d,2,r)$*, Differential Geometry and Its Applications (Eger, 1989), 317–357, Colloq. Math. Soc. János Bolyai **56**, North-Holland Publ. Co., Amsterdam, 1992.

[G 99]   Goldberg, V. V., *A classification and examples of isoclinic of four-dimensional three-webs* $W(d, 2, r)$, Webs & quasigroups, 32–66, Tver. Gos. Univ., Tver', 1998–1999.

[H 98]   Hénaut, A., *Sur l'algebrisation des tissus de codimension n de $\mathbf{C}^{2n}$* (French) [On algebraization problem for webs of codimension n in $\mathbf{C}^{2n}$], Ann. Sci. École Norm. Sup. **31** (1998), no. 1, 131–143.

Author's address:
Vladislav V. Goldberg
Department of Mathematics
New Jersey Institute of Technology
Newark, N.J. 07102, U.S.A.

Author's e-mail address: vlgold@m.njit.edu